Math 211

Ordinary Differential Equations and Linear Algebra
Solutions for Second Midterm Exam, Spring 2002

Rice University April 2, 2002

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1. Suppose you are given the differential equation $y' = y + 2t$.

   (a) (6 points) Showing all steps, using Euler's method with 2 steps (stepsize 0.5) find the numerical value of the solution at time $t = 1$ if $y(0) = -1$.

   Solution: Euler's method is repeated application of the tangent line approximation.
   At $t = 0$, the value of the derivative of the solution with value $y(0) = -1$ is $y'(0)$. From the differential equation

   $$y'(0) = -1 + 2(0) = -1$$

   Hence the numerical value of the solution at $t = 0.5$ is

   $$y(0.5) = y(0) + 0.5y'(0) = -1 - 0.5 = -1.5$$

   Repeating this procedure, $y'(0.5) = -1.5 + 2(0.5) = -0.5$. Then the numerical value of the solution at $t = 1$ is

   $$y(1) = y(0.5) + 0.5y'(0.5) = -1.5 - 0.25 = -1.75$$

   (b) (10 points) Solve the ODE analytically and find the actual value of $y(1)$ given the same initial condition $y(0) = -1$.

   Solution: This is a linear first order ODE with $a(t) = 1$ and $f(t) = 2t$. We will solve this using the method of integrating factors (you could also use variation of parameters).
   We have $u = u(t) = e^{-\int a(t)dt} = e^{-t}$. Then the solution is

   $$y = y(t) = \frac{1}{u}\int uf\,dt + \frac{C}{u}$$

   where C is a constant.
   Solving this gives $y(t) = -2t - 2 + Ce^t$. Using the initial condition $y(0) = -1$ gives $C = 1$ i.e. $y = y(t) = -2t - 2 + e^t$. Thus

   $$y(1) = -2(1) - 2 + e^1 = -4 + 2.7 = -1.3$$

   (c) (2 points) Find the total error in your answer from part (a) (given that $e = e^1 = 2.7$).

   Solution: The total error is: $|-1.75 - (-1.3)| = 0.45$.

(d) (2 points) In one sentence describe how to use Euler's method over the same interval with the same initial condition to get a more accurate answer.

Solution: One can make the error less by using more steps i.e. by making the step size smaller.

2. Consider the system of equations:

$$x + 2y + 3z = 4$$
$$x + y + z = 2$$
$$x + 3y + 5z = 6$$

(a) (2 points) Write down the augmented matrix associated to this system.

Solution: The augmented matrix is

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \\ 1 & 3 & 5 & 6 \end{pmatrix}$$

(b) (5 points) Find row-echelon form for the matrix in part (a).

Solution: Perform row operations:
$R1 \leftrightarrow R2$:

$$A = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 6 \end{pmatrix}$$

$R2 - R1$ and $R3 - R1$:

$$A = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 4 & 4 \end{pmatrix}$$

$R3 - 2R1$:

$$A = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is row-echelon form.

(c) (3 points) What is the dimension of the solution set to this system of equations? Explain in one sentence how to get this answer from the answer to part (b).

Solution: The dimension of the solution set is 1 (i.e. the solution is a line). This is the number of free variables in the solution which is the number of columns of the matrix in row-echelon form corresponding to unknowns without pivots.

(d) (5 points) Solve the system of equations.

Solution: From part (b) we see there are two pivot variables. Since there is no pivot in the last column (the RHS of the system) there is a solution to the system. Since there is no pivot in the third column, z is a free variable. Let $z = s$, s real.

Now back-solve the system. $R2$ says $y + 2z = 2$. Using $z = s$ and rearranging gives $y = 2 - 2s$.

$R1$ says $x + y + z = 2$. Using $y = 2 - 2s$ and $z = s$ and simplifying and rearranging gives $x = s$.

In total our solution is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, s \text{ real}$$

3. Consider the matrix

$$A = \begin{pmatrix} x & 1 \\ 1 & x \end{pmatrix}$$

(a) (3 points) Find the determinant of A in terms of x.

Solution: For a 2×2 matrix there is a very easy formula for the determinant. Applied to the matrix in this question we get $\det(A) = x^2 - 1$.

(b) (4 points) Find the value(s) of x that make A non-singular.

Solution: A being non-singular means $\det(A) \neq 0$. Thus we need to find the values of x that make $X^2 - 1 \neq 0$. This gives $x < 1$, $-1 < x < 1$ and $x > 1$.

(c) (4 points) Find the value(s) of x that ensure A has non-trivial nullspace.

Solution: A having non-trivial nullspace is the same as A being singular (for square matrices) which is the same as $\det(A) = 0$. Thus we need to solve $x^2 - 1 = 0$. Solving gives $x = -1$ or $x = 1$.

(d) (4 points) For $x = -1$ find the nullspace of A.

Solution: We need to find nullspace of the matrix:

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

This is the same as solving the system of equations

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Form the augmented matrix and adding R1 to R2 gives row-echelon form:

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

There is no pivot in the second column so y is a free variable; say $y = s, s$ real. Backsolving then gives: $-x + y = 0$ i.e. $x = s$.
In total the solution, which is the nullspace, is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \end{pmatrix}, s \text{ real}$$

Thus the nullspace is a line.

4. Consider the vectors

$$x_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, x_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, x_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(a) (6 points) Find the nullspace of the 2×3 matrix whose columns are the vectors x_1, x_2 and x_3.

Solution: We need to solve the system with corresponding augmented matrix:

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 1 & 0 \end{pmatrix}$$

R2-2R1 gives:

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -3 & -1 & 0 \end{pmatrix}$$

This is row-echelon form. There is no pivot in the third row so z is a free variable; say $z = s, s$ real. Backsolving gives: $-3y - z = 0$ i.e. $y = \frac{-s}{3}$. Then $x + 2y + z = 0$ i.e. $x = \frac{-s}{3}$.
In total the nullspace is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = s \begin{pmatrix} -1/3 \\ -1/3 \\ 1 \end{pmatrix}, s \text{ real}$$

(b) (2 points) Are the vectors x_1, x_2, x_3 linearly **independent**? Explain in one sentence.

Solutions: No, the vectors are linearly independent becuase the nullspace of the corresponding matrix (with the vectors as columns) is non-trivial.

(c) (2 points) Find a non-zero linear combination of the vectors x_1, x_2, x_3 equalling the zero vector.

Solutions: Any non-zero vector in the nullspace will suffice e.g. with $s = 3$ we get $-x_1 - x_2 + 3x_3$ is the zero vector.

(d) (3 points) What is the dimension of $span(x_1, x_2, x_3)$? Explain in one sentence how you got this dimension.

Solution: The dimension is 2. This is the number or vectors minus the dimension of the corresponding nullspace i.e. 2=3-1.

(e) (2 points) Write down a basis for $span(x_1, x_2, x_3)$.

Solution: A basis is a collection of linearly independent vectors with the same span. Since the nullspace has dimension 1 we need to throw away 1 vector to get linearly independent vectors. So we have three choices for a basis: $\{x_1, x_2\}$ or $\{x_1, x_3\}$ or $\{x_2, x_3\}$.

5. (10 points) Consider the differential equation $y'' + 3y' + 2y = 0$. Write down an equivalent first order system of differential equations. Show all steps in your derivation of this system.

Solution: This is a second order differential equation so the corresponding system of first order equations will have two unknowns.

Let $u_1 = y$ and $u_2 = y'$.

Then our first equation is $u_1' = u_2$ coming from the relationship between the new variables.

Using the new variables the original ODE becomes: $u_2' + 3u_2 + 2u_1 = 0$. Rearranging this gives: $u_2' = -2u_1 - 3u_2$.

Thus the new system of first order ODEs is:

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= -2u_1 - 3u_2 \end{aligned}$$

6. Consider the first order system of differential equations (independent variable t):

$$x' = 2x + 6y$$
$$y' = x + y$$

(a) (5 points) Does $x(t) = -2e^{-t}, y(t) = e^{-t}$ give a solution to this system? Show all calculations supporting your answer.

Solution: Need to see if the suggested functions make the equations in the system true. Calculate LHS and RHS of each equation.
For the first equation:
LHS$=x' = 2e^{-t}$
RHS$=2x + 6y = 2(-2e^{-t}) + 6(e^{-t}) = 2e^{-t}$
Since LHS=RHS the functions make this equation true.
Now check the second equation:
LHS$=y' = -e^{-t}$
RHS$=x + y = -2e^{-t} + e^{-t} = -e^{-t}$.
Since LHS=RHS the functions make this equation true also.
Since both equations are true for these functions they are a solution to the system.

(b) (5 points) Find the equations for the nullclines of this system.

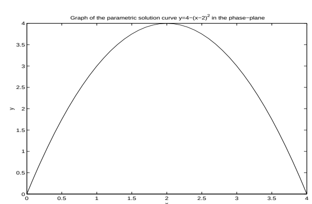
Solution: Nullclines for the first equation are when RHS=0 i.e. need to solve $2x+6y = 0$. Solving this gives a single nullcline $y = (-1/3)x$. Nullclines for the second equation are when RHS=0 i.e. need to solve $x + y = 0$. Solving gives a single nullcline $y = -x$.

(c) (5 points) Find all equilibrium points for this system.

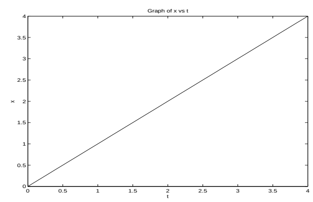
Solution: Equilibrium points can be found by finding all points of intersection of 2 nullclines, one from each equation. In this case equilibrium points will be points of intersection of the nullcline $y = (-1/3)x$ with the nullcline $y = -x$. In this case there is a single point of intersection $x = 0, y = 0$ i.e. the equilibrium point is $\binom{x}{y} = \binom{0}{0}$

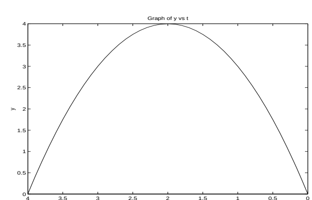
7. (10 points) For a planar system with unknowns x and y (independent variable t) suppose that a parametric solution curve is given by the curve $y = 4 - (x-2)^2$ in the phase-plane. Suppose that as time increases the solution moves along this curve from left to right. If at time $t = 0$ you start at the point $(x, y) = (0, 0)$ sketch solution curves x vs t and y vs t for this system from this parametric solution curve up until you reach the point $(x, y) = (4, 0)$.

Solution: Here is a picture of the parametric solution curve in the phase-plane:



In words, since we are travelling from left to right along this parametric solution curve from (0,0) to (4,0) the x-variable increases from 0 to 4 and the y variable increases from 0 to 4 and then decreases from 4 to 0 in the same time. When the y variable is at 4 the x variable has value 2. Making plots of this gives the following graphs (where the t-zxis assumes that it has taken time=4 units to go from (0,0) to (4,0); this scale could be different in your graphs):


Graph of x vs t
x
t
0
0.5
1
1.5
2
2.5
3
3.5
4


Graph of y vs t
y
t
0
0.5
1
1.5
2
2.5
3
3.5
4