Mathematics 211 Midterm 1, October 3, 2000

1. Find the solution of the initial-value problem given by

$$\frac{dy}{dt} = -\frac{y}{t} + 3t + 2, \qquad y(-1) = 2.$$

Also find the interval of existence.

2. Find the solution of the following initial-value problem and state its interval of existence:

$$\frac{dy}{dt} = \frac{2t}{y + yt^2}, \qquad y(0) = 2.$$

3. Chemists study the rate at which two substances combine to form a third. By applying the law of mass action, they find that, if x is the amount of the new compound, then the rate of change is

$$\frac{dx}{dt} = k(a - x)(b - x),$$

where k, a and b are constants with $k > 0$ and $b > a > 0$.

(a) Draw the phase line associated to this autonomous system, and discuss the stability of each equilibrium point.

(b) Discuss the behavior of $x(t)$ as $t \to \infty$ for the various values of the initial condition $x(0)$ between 0 and b.

4. A 5000-gallon tank contains 100 gallons of pure water. A salt solution containing 1 pound of salt per gallon flows into the tank at 2 gallons per minute. From another pipe a salt solution containing 2 pounds per gallon of salt enters at 2 gallons per minute. The well-mixed solution flows out at 3 gallons per minute. Find the concentration of the salt in the tank after 100 minutes.

More on the other side

5. A population of bacteria (measured in thousands) is governed by the equation $P' = P$, where time is measured in hours and P in thousands. At $t = 0$, a toxin is introduced killing off kt bacteria per hour at time t, where k is some positive constant.

   (a) Write down a differential equation modelling this modified Malthusian situation.

   (b) At $t = 2$, it is observed that $P = 60$ and the rate of growth is 20 bacteria per hour. Find the amount of bacteria initially in the sample.

6. An object of mass m falls from rest towards earth. It encounters an air resistance force R that depends on the velocity according to the experimentally discovered rule

$$R(v) = -kv(1 + a|v|),$$

   where k and a are positive constants. Find the terminal velocity of the object. Hint: It is not necessary to solve for the velocity as a function of t.

7. Consider the differential equation

$$\frac{dy}{dt} = \frac{t^2 - 4y - 3}{1 + y^2}.$$

   (a) Show that the function $y(t) = t - 2$ is a solution to this equation.

   (b) Let $\widetilde{y}(t)$ denote the solution of the above equation passing through $(0, 0)$. Show that $\lim_{t\to\infty} \widetilde{y}(t) = \infty$. Note: If you use any theorems studied in class, you must show carefully that all the hypotheses are satisfied.