Math 211

Ordinary Differential Equations and Linear Algebra
Second Midterm Exam, Spring 2002

Rice University April 2, 2002

Instructions: this is a 75 minute closed book exam. It is a pledged exam. Please write the pledge on your exam script. You may not use calculators at all. Please show all working. You may leave numerical answers unsimplified.

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1. Suppose you are given the differential equation $y' = y + 2t$.

   (a) (6 points) Showing all steps, using Euler's method with 2 steps (step-size 0.5) find the numerical value of the solution at time $t = 1$ if $y(0) = -1$.

   (b) (10 points) Solve the ODE analytically and find the actual value of $y(1)$ given the same initial condition $y(0) = -1$.

   (c) (2 points) Find the total error in your answer from part (a) (given that $e = e^1 = 2.7$).

   (d) (2 points) In one sentence describe how to use Euler's method over the same interval with the same initial condition to get a more accurate answer.

2. Consider the system of equations:

$$x + 2y + 3z = 4$$
$$x + y + z = 2$$
$$x + 3y + 5z = 6$$

   (a) (2 points) Write down the augmented matrix associated to this system.

   (b) (5 points) Find row-echelon form for the matrix in part (a).

   (c) (3 points) What is the dimension of the solution set to this system of equations? Explain in one sentence how to get this answer from the answer to part (b).

   (d) (5 points) Solve the system of equations.

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3. Consider the matrix

$$A = \begin{pmatrix} x & 1 \\ 1 & x \end{pmatrix}$$

   (a) (3 points) Find the determinant of A in terms of x.
   (b) (4 points) Find the value(s) of x that make A non-singular.
   (c) (4 points) Find the value(s) of x that ensure A has non-trivial nullspace.
   (d) (4 points) For $x = -1$ find the nullspace of A.

4. Consider the vectors

$$x_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, x_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, x_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

   (a) (6 points) Find the nullspace of the 2×3 matrix whose columns are the vectors x_1, x_2 and x_3.
   (b) (2 points) Are the vectors x_1, x_2, x_3 linearly **independent**? Explain in one sentence.
   (c) (2 points) Find a non-zero linear combination of the vectors x_1, x_2, x_3 equalling the zero vector.
   (d) (3 points) What is the dimension of $span(x_1, x_2, x_3)$? Explain in one sentence how you got this dimension.
   (e) (2 points) Write down a basis for $span(x_1, x_2, x_3)$.

5. (10 points) Consider the differential equation $y'' + 3y' + 2y = 0$. Write down an equivalent first order system of differential equations. Show all steps in your derivation of this system.

6. Consider the first order system of differential equations (independent variable t):

$$x' = 2x + 6y$$
$$y' = x + y$$

   (a) (5 points) Does $x(t) = -2e^{-t}, y(t) = e^{-t}$ give a solution to this system? Show all calculations supporting your answer.
   (b) (5 points) Find the equations for the nullclines of this system.
   (c) (5 points) Find all equilibrium points for this system.

7. (10 points) For a planar system with unknowns x and y (independent variable t) suppose that a parametric solution curve is given by the curve $y = 4 - (x - 2)^2$ in the phase-plane. Suppose that as time increases the solution moves along this curve from left to right. If at time $t = 0$ you start at the point $(x, y) = (0, 0)$ sketch solution curves x vs t and y vs t for this system from this parametric solution curve up until you reach the point $(x, y) = (4, 0)$.