

Math 211

Ordinary Differential Equations and Linear Algebra First Midterm Exam, Spring 2002

Rice University

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Instructions: this is a 75 minute closed book exam. It is a pledged exam. Please write the pledge on your exam script. You may not use calculators at all. Please show all working. You may leave numerical answers unsimplified.

1. Suppose you are given the differential equation $(t - 3)^2 y' = 3y$.
 - (a) (7 points) Is the function $y(t) = \frac{1}{(t-3)}$ a solution?
 - (b) (3 points) What is the value of the slope of the tangent line of a solution passing through the point $(t, y) = (4, 1)$?
 - (c) (15 points) Solve the given differential equation with initial condition $y(0) = 3e$ including the interval of existence (here e is the base of the natural logarithm i.e. $e = e^1 = 2.71828\dots$).

2. Suppose a mass of 4kg is moving under the influence of gravity (use $g = 10$ m/s²).
 - (a) (10 points) If the air-resistance is proportional to the velocity of the object with constant of proportionality r (where $r > 0$), find the formula for the velocity of the object for $t \geq 0$ if the velocity at time $t = 0$ seconds is $\frac{20}{r}$ m/s (NOTE: this formula will involve r).
 - (b) (5 points) Suppose you also know that the terminal velocity of the object is -10 m/s. Find the time at which the object is at its highest point.
 - (c) (5 points) Find the height of the object at the highest point if it starts out at a height of 15 metres above ground level (do not simplify your expression for the height).

3. Suppose a swimming pool of total volume 4000 litres initially has 1000 litres of a salt solution of concentration 2 kg/litre. Suppose salt solution of concentration 0.4 kg/litre is being poured into the pool at a rate of 10 litres/minute and salt solution is leaving the pool at a rate of 5 litres/minute. Assume that all solutions mix instantaneously.

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- (a) (5 points) Show that the differential equation modelling the total amount of salt, $A = A(t)$ (measured in kg) in the pool at time t (minutes) is given by:

$$\frac{dA}{dt} = 4 - A/(200 + t)$$

- (b) (12 points) Solve this differential equation to find a formula for $A = A(t)$.
- (c) (3 points) Find the total amount of salt in the pool at the time it becomes full.
4. Suppose a lake has a logistic model for the population of fish $P = P(t)$ (t =time in days) given by the equation $\frac{dP}{dt} = P(1 - P/600)$. Suppose also that people fish this lake and remove 10 percent of the fish population per day.
- (a) (2 points) Write a model for the population of fish in the lake.
- (b) (10 points) Find the phase-line for this model.
- (c) (4 points) Classify each equilibrium solution as stable or unstable.
- (d) (4 points) If at $t = 0$ the population is 200 fish, what value will the population of fish approach as $t \rightarrow +\infty$?

5. Suppose you are given the differential equation $y' = -e^{2t}y^3$.

- (a) (5 points) Find the values of K that make $y(t) = Ke^{-t}$ a solution to this equation (where K is a constant).
- (b) (10 points) If $y = y(t)$ is also a solution and $y(0) = 0.5$ what is $\lim_{t \rightarrow \infty} y(t)$? (NOTE: you don't necessarily need to solve the ODE to answer this question but you must justify your answer fully).