

Math 211
Final Exam

Spring 2001

Instructions: This is a closed book exam. Write out and sign the honor pledge on your exam book. In addition please print your name on your exam book.

You are allowed to use a calculator to do simple computations. You are **not** allowed to use a calculator for any symbolic computations such as computing derivatives or integrals, or to solve differential equations.

Please give reasons for all of your answers. There is a short table of integrals on the last page that might be of some help.

1. Consider the differential equation

$$y'' + 5y' + 4y = 0.$$

- a) (4 points) Find a fundamental set of solutions.
- b) (3 points) What is the general solution?
- c) (3 points) Find the solution satisfying $y(0) = 1$ and $y'(0) = 2$.

2. Consider the differential equation

$$x' = (1 + x^2) \cos t.$$

- a) (4 points) What is the general solution?
- b) (3 points) Find the solution satisfying $x(0) = 0$.

3. Consider the system

$$\begin{aligned}x' &= y + (1 - x^2 - y^2) \\y' &= -x + (1 - x^2 - y^2)\end{aligned}$$

- a) (4 points) Show that the pair $x(t) = \sin t$, $y(t) = \cos t$ is a solution.
- b) (3 points) Show that the unit circle defined by $x^2 + y^2 = 1$ is invariant.
- c) (3 points) Show that the interior of the unit circle, defined by $x^2 + y^2 < 1$ is invariant.
- d) (3 points) Is the outside of the unit circle, defined by $x^2 + y^2 > 1$ is invariant?

e) (3 points) What can you say about the type of the equilibrium point at the origin $(0, 0)^T$?

f) (3 points) Describe what happens to all solutions starting away from the origin. (*Hint:* Look at the function $r(t) = \sqrt{x^2(t) + y^2(t)}$.)

4. (7 points) Consider two species with populations $x(t)$ and $y(t)$ that live in the same area, and interact in the following ways:

- In the absence of the other population, population x would thrive subject to limited resources.
- In the absence of the other population, population y would die out.
- The two populations have a symbiotic relationship, meaning that each benefits from the presence of the other.

Derive a differential equation model of the interaction of these two species. It is not necessary to solve the equations.

5. Consider the system

$$\begin{aligned}x' &= y - x^2 \\y' &= x + y\end{aligned}$$

a) (4 points) Find the nullclines and sketch them, indicating clearly which is the x -nullcline and which is the y -nullcline.

b) (3 points) Find all of the equilibrium points.

c) (3 points) Compute the Jacobian matrix.

d) (3 points) Classify the equilibrium points.

e) (3 points) Draw arrows on the nullclines indicating the direction of the solution curves.

f) (3 points) The nullclines divide the plane into several regions. Which of these regions are invariant?

6. Consider the following equation for a forced harmonic oscillator.

$$y'' + 4y' + 8y = 20 \cos 2t.$$

a) (4 points) What is the associated homogeneous equation?

b) (3 points) Find the general solution for the homogeneous equation.

c) (3 points) Find a particular solution to the inhomogeneous equation.

d) (3 points) Find the general solution for the inhomogeneous equation.

- e) (3 points) What is the steady-state solution?
- f) (3 points) What is the amplitude of the steady-state solution?

7. Consider the system of equations $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 9 \\ 0 & -1 & -5 \end{pmatrix}.$$

- a) (4 points) What is the characteristic polynomial of A ?
- b) (3 points) What are the eigenvalues of A , and what are the algebraic and geometric multiplicities of each?
- c) (3 points) For each eigenvalue, find a basis of the eigenspace.
- d) (3 points) Find a fundamental set of solutions for the system.
- e) (3 points) Find the solution which satisfies the initial condition $\mathbf{y}(0) = (1, 2, 0)^T$.
- f) (3 points) What can you say about the type of the equilibrium point at the origin $(0, 0, 0)^T$?

Table of Integrals

The letters a , b , c , and d stand for arbitrary constants.

$$\int \sec ax \, dx = \frac{1}{a} \ln \tan \left(\frac{\pi}{4} + \frac{ax}{2} \right)$$

$$\int \tan ax \, dx = -\frac{1}{a} \ln \cos ax$$

$$\int \cot ax \, dx = \frac{1}{a} \ln \sin ax$$

$$\int \frac{ay + b}{cy + d} \, dy = \frac{ay}{c} + \frac{bc - ad}{c^2} \ln(cy + d)$$

$$\int \frac{dx}{a^2 + b^2x^2} = \frac{1}{ab} \arctan \left(\frac{bx}{a} \right)$$

$$\int \frac{y}{(ay + b)(cy + d)} \, dy = \frac{1}{bc - ad} \left[\frac{b}{a} \ln(ay + b) - \frac{d}{c} \ln(cy + d) \right]$$

$$\int \frac{dy}{(ay + b)(cy + d)} = \frac{1}{bc - ad} \ln \left(\frac{cy + d}{ay + b} \right)$$