Math 211

Final Exam

April 25, 2000

Instructions: This is a closed book, three hour exam. You are allowed to use a calculator to do simple computations. You are **not** allowed to use a calculator for any symbolic computations such as computing derivatives or integrals, or to solve differential equations.

Please give reasons for all of your answers.

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1. (5 points) Solve the initial value problem

$$\frac{dy}{dt} = (1+t)\sin t + \frac{y}{1+t} \quad \text{with} \quad y(0) = 2.$$

Answer: This equation is linear, with the coefficient of y being $a(t) = (1+t)^{-1}$. An integrating factor is

$$\begin{aligned} u(t) &= e^{-\int a(t)\,dt} \\ &= e^{-\int \frac{dt}{1+t}} \\ &= e^{-\ln(1+t)} \\ &= \frac{1}{1+t}. \end{aligned}$$

Then we have

$$\left(\frac{y}{1+t}\right)' = \frac{1}{1+t}\left(y' - \frac{y}{1+t}\right) = \sin t.$$

Integrating we get

$$\frac{y}{1+t} = -\cos t + C \quad \text{or} \quad y(t) = (1+t)(C - \cos t).$$

The initial condition enables us to compute that

$$2 = y(0) = C - 1 \quad \text{or} \quad C = 3.$$

Hence the solution to the initial value problem is

$$y(t) = (1+t)(3 - \cos t).$$

2. (5 points) Solve the initial value problem

$$\frac{dy}{dx} = 1 + y + t^2 y + t^2 \quad \text{with} \quad y(0) = 0.$$

Answer: The differential equation is

$$\frac{dy}{dx} = 1 + y + t^2 y + t^2 = (1 + y)(1 + t^2),$$

so the equation is separable. In the usual way we write

$$\frac{dy}{1 + y} = (1 + t^2)\, dt.$$

Integrating both sides yields

$$\ln(1 + y) = t + t^3/3 + C.$$

Solving for y,

$$y(t) = e^{t + t^3/3 + C} - 1.$$

The intial condition tells us that

$$0 = y(0) = e^C - 1 \quad \text{or} \quad C = 0.$$

Hence the solution is

$$y(t) = e^{t + t^3/3} - 1.$$

3. (5 points) Find the general solution of the system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}.$$

What is the type of the equilibrium point that this system has at the origin?

Answer: The characteristic polynomial is

$$\begin{aligned} p(\lambda) = \det(A - \lambda I) &= \det \begin{pmatrix} -4-\lambda & 6 \\ -3 & 5-\lambda \end{pmatrix} \\ &= (-4-\lambda)(5-\lambda) + 18 = \lambda^2 - \lambda - 2 \\ &= (\lambda+1)(\lambda-2). \end{aligned}$$

Thus the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 2$. Corresponding to $\lambda_1 = -1$ we have

$$(A - \lambda_1 I) = (A + I) = \begin{pmatrix} -3 & 6 \\ -3 & 6 \end{pmatrix}.$$

Clearly $\mathbf{v}_1 = (2, 1)^T$ is an eigenvector, and the corresponding solution is

$$\mathbf{y}_1(t) = e^{\lambda_1 t}\mathbf{v}_1 = e^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Corresponding to $\lambda_2 = 2$ we have

$$(A - \lambda_2 I) = (A - 2I) = \begin{pmatrix} -6 & 6 \\ -3 & 3 \end{pmatrix}.$$

Clearly $\mathbf{v}_2 = (1, 1)^T$ is an eigenvector, and the corresponding solution is

$$\mathbf{y}_2(t) = e^{\lambda_2 t}\mathbf{v}_2 = e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The general solution is

$$\mathbf{y}(t) = C_1\mathbf{y}_1(t) + C_2\mathbf{y}_2(t) = C_1 e^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Since the eigenvalues have different signs, the origin is a saddle point.

4. (5 points) Solve the initial value problem

$$x'' + 6x' + 13x = 0, \quad \text{with} \quad x(0) = 2 \quad \text{and} \quad x'(0) = 0.$$

Answer: The characteristic polynomial is $p(\lambda) = \lambda^2 + 6\lambda + 13$. It has roots

$$\begin{aligned} \lambda &= \frac{1}{2}\left[-6 \pm \sqrt{36 - 52}\right] \\ &= -3 \pm 2i \end{aligned}$$

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Hence the general solution is

$$x(t) = e^{-3t}\left(C_1 \cos 2t + C_2 \sin 2t\right).$$

From the first initial condition we get

$$2 = x(0) = C_1.$$

Differentiating, we find that

$$x'(t) = -3e^{-3t}\left(C_1 \cos 2t + C_2 \sin 2t\right) + 2e^{-3t}\left(-C_1 \sin 2t + C_2 \cos 2t\right).$$

Then from the second initial condition we get

$$0 = x'(0) = -3C_1 + 2C_2 = -6 + 2C_2 \quad \text{or} \quad C_2 = 3.$$

The solution to the initial value problem is

$$x(t) = e^{-3t}\left(2 \cos 2t + 3 \sin 2t\right).$$

5. (5 points) Solve the initial value problem

$$x'' - 4x = (9t - 3)e^t, \quad \text{with} \quad x(0) = 0 \quad \text{and} \quad x'(0) = 0.$$

Answer: The homogeneous eqution is $x'' - 4x = 0$, which has general solution

$$x_h(t) = C_1 e^{2t} + C_2 e^{-2t}.$$

We need to find a particular solution x_p to the inhomogeneous equation. To do so we will use undetermined coefficients. We propose that x_p has the same form as the forcing term. Hence we have

$$\begin{aligned} x_p(t) &= (at + b)e^t \\ x_p'(t) &= (at + b + a)e^t \quad \text{and} \\ x_p''(t) &= (at + b + 2a)e^t \end{aligned}$$

Hence

$$\begin{aligned} x_p'' - 4x_p &= [(at + b + 2a) - 4(at + b)]e^t \\ &= [-3at + (2a - 3b)]e^t. \end{aligned}$$

For this to be a solution we must have

$$-3a = 9 \quad \text{and} \quad 2a - 3b = -3.$$

Hence $a = -3$ and $b = -1$. Thus $x_p(t) = -(3t + 1)e^t$, and the general solution to the inhomogeneous equaiton is

$$\begin{aligned} x(t) &= x_p(t) + x_h(t) \\ &= -(3t + 1)e^t + C_1 e^{2t} + C_2 e^{-2t}. \end{aligned}$$

By differentiation we see that

$$x'(t) = -(3t + 4)e^t + +2C_1 e^{2t} - 2C_2 e^{-2t}.$$

The initial conditions become

$$\begin{aligned} 0 &= x(0) = -1 + C_1 + C_2 \\ 0 &= x'(0) = -4 + C_1 - C_2 \end{aligned}$$

Solving we obtain $C_1 = 5/2$ and $C_2 = -3/2$, so the desired solution is

$$x(t) = -(3t + 1)e^t + \frac{5}{2}e^{2t} - \frac{3}{2}e^{-2t}.$$

6. Consider the non-linear equation

$$x'' + 4x - x^3 = 0.$$

a) (5 points) What is the associated first order system?

Answer: If we introduce $y = x'$, then the first order system is

$$\begin{aligned} x' &= y \\ y' &= x^3 - 4x. \end{aligned}$$

b) (5 points) Find all critical points for the associated first order system.

Answer: We need $y = 0$ and $x^3 - 4x = 0$. Since $x^3 - 4x = x(x-2)(x+2)$, we must have $x = -2,\ 0,$ or 2. Thus there are three equilibrium points, $(-2, 0)$, $(0, 0)$, and $(2, 0)$.

c) (5 points) Describe the behaviour of the linearized equation at each of the critical points.

Answer: The Jacobian matrix is

$$J = \begin{pmatrix} 0 & 1 \\ 3x^2 - 4 & 0 \end{pmatrix}.$$

At $(0, 0)$,

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}.$$

The determinant is $D = 4$ and the trace is $T = 0$. This equilibrium point is a center.

At $(\pm 2, 0)$,

$$J(\pm 2, 0) = \begin{pmatrix} 0 & 1 \\ 8 & 0 \end{pmatrix}.$$

Now the determinant is $D = -8$ and the trace is $T = 0$. These equilibrium points are saddle points.

d) (5 points) On the basis of the linear analysis in part c), what can you say about the nature of the solutions to the nonlinear system near each of the critical points?

Answer: Saddles are generic, so the nonlinear system has saddle points at $(\pm 2, 0)$. However centers are not generic, so the fact that the linearization at the origin has a center, does not mean that the nonlinear system has a center at the origin.

7. Consider the differential equation

$$y'' + 2by' + cy = 0,$$

where $b \neq 0$ and c are constants.

a) (5 points) Write down the first order system $\mathbf{x}' = A\,\mathbf{x}$ which is equivalent to this equation.

Answer: Setting $x_1 = y$ and $x_2 = y'$, the system is

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -cx_1 - 2bx_2. \end{aligned}$$

In the matrix form $\mathbf{x}' = A\,\mathbf{x}$, we have the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -c & -2b \end{pmatrix}.$$

b) (7 points) Suppose that the characteristic polynomial for the second order equation has a multiple root. Under this assumption, show that the matrix A has an eigenvalue of multiplicity 2, and that this eigenvalue is deficient; i.e., show that the geometric multiplicity of the eigenvalue is less than its algebraic multiplicity.

Answer: The characteristic polynomial for the second order equation is the same as that for the matrix A, and is $p(\lambda) = \lambda^2 + 2b\lambda + c$. This polynomial has roots

$$\lambda = -b \pm \sqrt{b^2 - c}.$$

There is a double root when $b^2 - c = 0$. In this case $c = b^2$, and

$$A = \begin{pmatrix} 0 & 1 \\ -b^2 & -2b \end{pmatrix}.$$

The only eigenvalue for A is $\lambda = -b$. It has algebraic multiplicity 2. On the other hand,

$$A - \lambda I = A + bI = \begin{pmatrix} b & 1 \\ -b^2 & -b \end{pmatrix}.$$

The dimension of the nullspace of $A - \lambda I$ is 1, and it is spanned by the eigenvector

$$\mathbf{v} = \begin{pmatrix} 1 \\ -b \end{pmatrix}.$$

Hence the geometric multiplicity of A is 1.

8. A singer is trying to shatter a wine glass. The vibration of the glass is modeled by the equation

$$y'' + 2y' + 4y = 8\cos(2t),$$

where y is the displacement of the top of the glass (perhaps measured in millimeters). If $|x(t)|$ ever exceeds 7 the glass shatters.

a) (5 points) Show that the general solution to the homogeneous equation is

$$e^{-t}(C_1 \cos \sqrt{3}t + C_2 \sin \sqrt{3}t).$$

Answer: The characteristic polynomial for the homogeneous equation is $p(\lambda) = \lambda^2 + 2\lambda + 4$. It's roots are

$$\begin{aligned}\lambda &= \frac{1}{2}\left[-2 \pm \sqrt{4 - 4 \times 4}\right] \\ &= -1 \pm i\sqrt{3}.\end{aligned}$$

From this it follows immediately that the general solution to the homogenous equation is

$$y_h(t) = e^{-t}(C_1 \cos \sqrt{3}t + C_2 \sin \sqrt{3}t).$$

b) (5 points) What is the general solution to the inhomogeneous equation.

Answer: We need to find a particular solution to the inhomogeneous equation. We will use the complex method of undetermined coefficients, looking for a solution to the complex equation

$$z'' + 2z + 4z = 8e^{2it}$$

of the form

$$z(t) = ae^{2it}.$$

Then

$$z'' + 2z + 4z = [-4a + 4ia + 4a]\, e^{2it} = 4iae^{2it}.$$

The function z will be a solution if $4ia = 8$, or if $a = -2i$. Thus

$$\begin{aligned}z(t) &= -2ie^{2it} \\ &= 2\sin 2t - 2i\cos 2t.\end{aligned}$$

Our particular solution is $y_p(t) = \text{Real} z(t) = 2\sin 2t$. The general solution to the inhomogeneous equation is

$$y(t) = y_p(t) + y_h(t) = 2\sin 2t + e^{-t}\left(C_1 \cos \sqrt{3}t + C_2 \sin \sqrt{3}t\right).$$

c) (5 points) What is the solution to the inhomogeneous equation with initial conditions

$$y(0) = 0 \quad \text{and} \quad y'(0) = 0?$$

Answer: The first initial condition gives

$$0 = y(0) = C_1.$$

Putting this into the formula for y and differentiating we get

$$y'(t) = 4\cos 2t + e^{-t}\left[-C_2 \sin\sqrt{3}t + C_2\sqrt{3}\cos\sqrt{3}t\right].$$

Hence the second initial condition is

$$0 = y'(0) = 4 + C_2\sqrt{3}, \quad \text{so} \quad C_2 = -4/\sqrt{3}.$$

The solution to the initial value problem is

$$y(t) = 2\sin 2t - \frac{4}{\sqrt{3}}e^{-t}\sin\sqrt{3}t.$$

d) (5 points) Describe the long term behavior of the solution in part c).

Answer: The exponentil term in the solution tends to 0, so over time the solution tends to the steady state solution $y_p(t) = 2\sin 2t$.

e) (4 points) Will the glass shatter under the motion in the solution to part c)?

Answer: No. The solution satisfies

$$\begin{aligned}
|y(t)| &= |2\sin 2t - \frac{4}{\sqrt{3}}e^{-t}\sin\sqrt{3}t| \\
&\leq |2\sin 2t| + |\frac{4}{\sqrt{3}}e^{-t}\sin\sqrt{3}t| \\
&\leq 2 + \frac{4}{\sqrt{3}} \\
&< 2 + 4 = 6 < 8.
\end{aligned}$$

9. Consider the matrix

$$A = \begin{pmatrix} 2 & a & a \\ 2 & -3 & a \\ 3 & 2 & 0 \end{pmatrix}.$$

a) (5 points) Find a non-zero value of a for which A has determinant 0.

Answer: Expanding $\det A$ by the third row, we get

$$\begin{aligned} \det A &= 3 \det \begin{pmatrix} a & a \\ -3 & a \end{pmatrix} - 2 \det \begin{pmatrix} 2 & a \\ 2 & a \end{pmatrix} \\ &= 3(a^2 + 3a) + 0 \\ &= 3a(a+3). \end{aligned}$$

Hence $\det A = 0$ if $a = -3$.

b) (5 points) For this value of a find the nullspace of A.

Answer: With $a = -3$ we have

$$A = \begin{pmatrix} 2 & -3 & -3 \\ 2 & -3 & -3 \\ 3 & 2 & 0 \end{pmatrix}.$$

There are several ways to proceed. Let's choose a non-standard way. If we subtract row 1 of A from row 2, we get

$$\begin{pmatrix} 2 & -3 & -3 \\ 0 & 0 & 0 \\ 3 & 2 & 0 \end{pmatrix}.$$

The homogeneous system corresponding to this matrix is

$$\begin{aligned} 2x - 3y - 3z &= 0 \\ 3x + 2y &= 0. \end{aligned}$$

The second equation implies that $y = -3x/2$. Substituting into the first equation we get $2x + 9x/2 - 3z = 0$, or $z = 13x/6$. We will let $x = 6t$, where t is free. Then $z = 13x/6 = 13t$, and $y = -3x/2 = -9t$. Consequently the nullspace consists of all vectors of the form

$$t \begin{pmatrix} 6 \\ -9 \\ 13 \end{pmatrix}.$$

10. (9 points) Consider a system of three different species existing together. Population 3 depends on population 2 for its food supply, and it would die out if population 2 were not present. Populations 1 and 3 do not interact in any way. There are no dependencies between them. On the other hand, populations 1 & 2 depend on and compete for the same resources for food and space. An increase in population 1 would mean less resources for population 2 and vice versa. Derive a differential equation model for this system. This means that you should write down a system of differential equations which describes this phenomenon. (You are not required to solve the system or to analyze it.)

Answer: Let x_1, x_2, and x_3 denote populations 1, 2 & 3 respectively. Then x_1 interacts only with x_2, and the presence of x_2 decreases the reproductive rate for x_1. Assuming that x_1 itself satisfies the logistic model we have

$$x_1' = ax_1 - bx_1^2 - cx_1x_2.$$

The population x_2 is affected by competition with x_1 and predation by x_3. If we again assume that x_2 all by itself is governed by a logistic limit the equation for x_2 is

$$x_2' = Ax_2 - Bx_2^2 - Cx_1x_2 - Dx_3x_2.$$

For the predator population x_3 we have a negative reproductive rate in the absebce of x_2, so the equation is

$$x_3' = -\alpha x_3 + \beta x_2x_3.$$

All of the constants are positive under the assumption of logistic limits on x_1 and x_2. However, the problem does not state that this is assumed. An answer not contining b and B is acceptable.