10. (9 points) Consider a system of three different species existing together. Population 3 depends on population 2 for its food supply, and it would die out if population 2 were not present. Populations 1 and 3 do not interact in any way. There are no dependencies between them. On the other hand, populations 1 & 2 depend on and compete for the same resources for food and space. An increase in population 1 would mean less resources for population 2 and vice versa. Derive a differential equation model for this system. This means that you should write down a system of differential equations which describes this phenomenon. (You are not required to solve the system or to analyze it.)

9. Consider the matrix

$$A = \begin{pmatrix} 2 & a & a \\ 2 & -3 & a \\ 3 & 2 & 0 \end{pmatrix}.$$

a) (5 points) Find a non-zero value of a for which A has determinant 0.

b) (5 points) For this value of a find the nullspace of A.

c) (5 points) What is the solution to the inhomogeneous equation with initial conditions

$$y(0) = 0 \quad \text{and} \quad y'(0) = 0?$$

d) (5 points) Describe the long term behavior of the solution in part c).

e) (4 points) Will the glass shatter under the motion in the solution to part c)?

8. A singer is trying to shatter a wine glass. The vibration of the glass is modeled by the equation

$$y'' + 2y' + 4y = 8\cos(2t),$$

where y is the displacement of the top of the glass (perhaps measured in millimeters). If $|x(t)|$ ever exceeds 7 the glass shatters.

a) (5 points) Show that the general solution to the homogeneous equation is

$$e^{-t}(C_1 \cos\sqrt{3}t + C_2 \sin\sqrt{3}t).$$

b) (5 points) What is the general solution to the inhomogeneous equation.

7. Consider the differential equation

$$y'' + 2by' + cy = 0,$$

where $b \neq 0$ and c are constants.

a) (5 points) Write down the first order system $\mathbf{x}' = A\,\mathbf{x}$ which is equivalent to this equation.

b) (7 points) Suppose that the characteristic polynomial for the second order equation has a multiple root. Under this assumption, show that the matrix A has an eigenvalue of multiplicity 2, and that this eigenvalue is deficient; i.e., show that the geometric multiplicity of the eigenvalue is less than its algebraic multiplicity.

6. Consider the non-linear equation

$$x'' + 4x - x^3 = 0.$$

a) (5 points) What is the associated first order system?

b) (5 points) Find all critical points for the associated first order system.

c) (5 points) Describe the behaviour of the linearized equation at each of the critical points.

d) (5 points) On the basis of the linear analysis in part c), what can you say about the nature of the solutions to the nonlinear system near each of the critical points?

5. (5 points) Solve the initial value problem

$$x'' - 4x = (9t - 3)e^t, \quad \text{with} \quad x(0) = 0 \quad \text{and} \quad x'(0) = 0.$$

4. (5 points) Solve the initial value problem

$$x'' + 6x' + 13x = 0, \quad \text{with} \quad x(0) = 2 \quad \text{and} \quad x'(0) = 0.$$

3. (5 points) Find the general solution of the system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}.$$

What is the type of the equilibrium point that this system has at the origin?

2. (5 points) Solve the initial value problem

$$\frac{dy}{dx} = 1 + y + t^2 y + t^2 \quad \text{with} \quad y(0) = 0.$$

Math 211

Final Exam

April 25, 2000

Instructions: This is a closed book, three hour exam. You are allowed to use a calculator to do simple computations. You are **not** allowed to use a calculator for any symbolic computations such as computing derivatives or integrals, or to solve differential equations.

Please give reasons for all of your answers.

Print your name:

Write out and sign the pledge:

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1. (5 points) Solve the initial value problem

$$\frac{dy}{dt} = (1+t)\sin t + \frac{y}{1+t} \quad \text{with} \quad y(0) = 2.$$