

Math 211

Final Exam — Answers

Spring 2001

1. Consider the differential equation

$$y'' + 5y' + 4y = 0.$$

a) (4 points) Find a fundamental set of solutions.

Answer: The characteristic polynomial is $p(\lambda) = \lambda^2 + 5\lambda + 4 = (\lambda + 1)(\lambda + 4)$. The characteristic roots are -1 and -4 , so the functions

$$y_1(t) = e^{-t} \quad \text{and} \quad y_2(t) = e^{-4t}$$

form a fundamental set of solutions.

b) (3 points) What is the general solution?

Answer: The general solution is

$$y(t) = C_1 y_1(t) + C_2 y_2(t) = C_1 e^{-t} + C_2 e^{-4t}.$$

c) (3 points) Find the solution satisfying $y(0) = 1$ and $y'(0) = 2$.

Answer: We need to find C_1 and C_2 so that

$$\begin{aligned} 1 &= y(0) = C_1 + C_2 \\ 2 &= y'(0) = -C_1 - 4C_2 \end{aligned}$$

These equations are solved by $C_1 = 2$ and $C_2 = -1$. The solution is

$$y(t) = 2e^{-t} - e^{-4t}.$$

2. Consider the differential equation

$$x' = (1 + x^2) \cos t.$$

a) (4 points) What is the general solution?

Answer: The equation is separable. Separating variables and then integrating we get

$$\int \frac{dx}{1+x^2} = \int \cos t dt$$
$$\arctan x = -\sin t + C$$
$$x(t) = \tan(C - \sin t).$$

b) (3 points) Find the solution satisfying $x(0) = 0$.

Answer: We have $0 = x(0) = \tan C$, so $C = 0$, and the solution is

$$x(t) = \tan(-\sin t) = -\tan(\sin t).$$

3. Consider the system

$$x' = y + (1 - x^2 - y^2)$$
$$y' = -x + (1 - x^2 - y^2)$$

a) (4 points) Show that the pair $x(t) = \sin t$, $y(t) = \cos t$ is a solution.

Answer: We have

$$x'(t) = \cos t \quad \text{and}$$
$$y + (1 - x^2 - y^2) = \cos t + (1 - \cos^2 t - \sin^2 t)$$
$$= \cos t$$

so the first equation is satisfied. For the second equation

$$y' = -\sin t \quad \text{and}$$
$$-x + (1 - x^2 - y^2) = -\sin t + (1 - \cos^2 t - \sin^2 t)$$
$$= -\sin t,$$

so it is satisfied as well.

b) (3 points) Show that the unit circle defined by $x^2 + y^2 = 1$ is invariant.

Answer: The unit circle consists of the solution curve found in part a). Since solution curves are invariant, the unit circle is invariant.

c) (3 points) Show that the interior of the unit circle, defined by $x^2 + y^2 < 1$ is invariant.

Answer: For a solution curve to leave the interior of the unit circle, it would have to cross the circle itself. Since the unit circle is a solution curve this would contradict the uniqueness theorem.

d) (3 points) Is the outside of the unit circle, defined by $x^2 + y^2 > 1$ is invariant?

Answer: Yes, the outside is invariant for the same reason.

e) (3 points) What can you say about the type of the equilibrium point at the origin $(0, 0)^T$?

Answer: The Jacobian of the system is

$$J(x, y) = \begin{pmatrix} -2x & 1 - 2y \\ -1 - 2x & -2y \end{pmatrix}.$$

At the origin we have

$$J(0, 0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This matrix has trace $T = 0$ and determinant $D = 1$. Consequently the origin is a center for the linearization. Since the center is not generic, we can conclude nothing about the equilibrium point for the nonlinear system.

f) (3 points) Describe what happens to all solutions starting away from the origin. (*Hint:* Look at the function $r(t) = \sqrt{x^2(t) + y^2(t)}$.)

Answer: We have

$$\begin{aligned} rr' &= xx' + yy' \\ &= x[y + (1 - x^2 - y^2)] + y[-x + (1 - x^2 - y^2)] \\ &= (x + y)(1 - r^2). \end{aligned}$$

However, this does not help much beyond confirming that $r = 1$ is a solution curve. Everyone gets 3 points for this one.

4. (7 points) Consider two species with populations $x(t)$ and $y(t)$ that live in the same area, and interact in the following ways:

- In the absence of the other population, population x would thrive subject to limited resources.
- In the absence of the other population, population y would die out.
- The two populations have a symbiotic relationship, meaning that each benefits from the presence of the other. Derive a differential equation model of the interaction of these two species. It is not necessary to solve the equations.

Answer: This is an instance of two interacting species. It leads to the equations

$$\begin{aligned} x' &= (a_1 - b_1x + c_1y)y \\ y' &= (-a_2 + c_2x)y, \end{aligned}$$

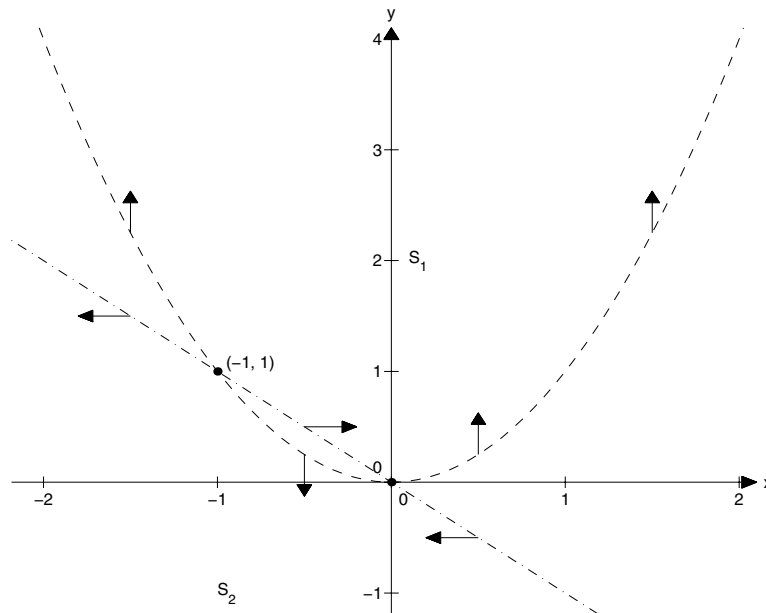
where all of the constants are positive.

5. Consider the system

$$\begin{aligned}x' &= y - x^2 \\y' &= x + y\end{aligned}$$

- a) (4 points) Find the nullclines and sketch them, indicating clearly which is the x -nullcline and which is the y -nullcline.

Answer: The x -nullcline is the parabola $y = x^2$. It is the dashed curve in the following figure. The y -nullcline is the line defined by $x + y = 0$, and it is dot-dashed in the figure.



- b) (3 points) Find all of the equilibrium points.

Answer: The equilibrium points are where the nullclines cross. There are two, at the origin $(0, 0)$ and at $(-1, 1)$.

- c) (3 points) Compute the Jacobian matrix.

Answer: The Jacobian is

$$J = \begin{pmatrix} -2x & 1 \\ 1 & 1 \end{pmatrix}.$$

- d) (3 points) Classify the equilibrium points.

Answer: At the origin the Jacobian is

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Since the determinant of J is -1 , the origin is a saddle point. At $(-1, 1)$ the Jacobian is

$$J = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

This time J has trace $T = 3$, and determinant $D = 1$. The discriminant is $T^2 - 4D = 5$. Since this is positive, the equilibrium point is a nodal source.

e) (3 points) Draw arrows on the nullclines indicating the direction of the solution curves.

Answer: The arrows are shown on the preceding figure.

f) (3 points) The nullclines divide the plane into several regions. Which of these regions are invariant?

Answer: There are two invariant regions. They are labelled S_1 and S_2 in the figure.

6. Consider the following equation for a forced harmonic oscillator.

$$y'' + 4y' + 8y = 20 \cos 2t.$$

a) (4 points) What is the associated homogeneous equation?

Answer: $y'' + 4y' + 8y = 0$.

b) (3 points) Find the general solution for the homogeneous equation.

Answer: The characteristic polynomial is $p(\lambda) = \lambda^2 + 4\lambda + 8$. the characteristic roots are

$$\begin{aligned} \lambda &= \frac{1}{2} \left(-4 \pm \sqrt{-16} \right) \\ &= \frac{1}{2} (-4 \pm 4i) \\ &= -2 \pm 2i. \end{aligned}$$

Hence the general solution is

$$y_h(t) = e^{-2t} [C_1 \cos 2t + C_2 \sin 2t].$$

c) (3 points) Find a particular solution to the inhomogeneous equation.

Answer: We will use the complex method and look for a solution of the form $z(t) = ae^{2it}$ to the equation $z'' + 4z' + 8z = 20e^{2it}$. Substituting, we get $z'' + 4z' + 8z = (4 + 8i)ae^{2it}$. We have a

solution if $(4 + 8i)a = 20$. Solving for a we get

$$\begin{aligned} a &= \frac{20}{4 + 8i} \\ &= \frac{5}{1 + 2i} \\ &= \frac{5}{1 + 2i} \frac{1 - 2i}{1 - 2i} \\ &= \frac{5(1 - 2i)}{1 + 4} \\ &= 1 - 2i. \end{aligned}$$

Therefore the complex solution is

$$\begin{aligned} z(t) &= (1 - 2i)e^{2it} \\ &= (1 - 2i)[\cos 2t + i \sin 2t] \\ &= [\cos 2t + 2 \sin 2t] + i[\sin 2t - 2 \cos 2t], \end{aligned}$$

and our particular solution is

$$\begin{aligned} y_p(t) &= \operatorname{Re}(z(t)) \\ &= \cos 2t + 2 \sin 2t. \end{aligned}$$

d) (3 points) Find the general solution for the inhomogeneous equation.

Answer: The general solution to the inhomogeneous equation is the sum of the particular solution and the general solution to the homogeneous equation. In this case that is

$$\begin{aligned} y(t) &= y_p(t) + y_h(t) \\ &= \cos 2t + 2 \sin 2t + e^{-2t} [C_1 \cos 2t + C_2 \sin 2t]. \end{aligned}$$

e) (3 points) What is the steady-state solution?

Answer: The steady-state solution is that part of the solution that does not die out. It is

$$y_p(t) = \cos 2t + 2 \sin 2t.$$

f) (3 points) What is the amplitude of the steady-state solution?

Answer: The amplitude of $y_p(t) = \cos 2t + 2 \sin 2t$ is

$$\sqrt{1^2 + 2^2} = \sqrt{5}.$$

7. Consider the system of equations $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 9 \\ 0 & -1 & -5 \end{pmatrix}.$$

a) (4 points) What is the characteristic polynomial of A ?

Answer: The characteristic polynomial is

$$p(\lambda) = \det \begin{pmatrix} -1 - \lambda & 0 & 0 \\ 1 & 1 - \lambda & 9 \\ 0 & -1 & -5 - \lambda \end{pmatrix}.$$

Expanding along the first row we get

$$\begin{aligned} p(\lambda) &= -(\lambda + 1) \det \begin{pmatrix} 1 - \lambda & 9 \\ -1 & -5 - \lambda \end{pmatrix} \\ &= -(\lambda + 1)[(\lambda - 1)(\lambda + 5) + 9] \\ &= -(\lambda + 1)[\lambda^2 + 4\lambda + 4] \\ &= -(\lambda + 1)(\lambda + 2)^2. \end{aligned}$$

b) (3 points) What are the eigenvalues of A , and what are the algebraic and geometric multiplicities of each?

Answer: The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = -2$. λ_1 has algebraic multiplicity 1 and therefore geometric multiplicity 1 as well. λ_2 has algebraic multiplicity 2. To discover its geometric multiplicity we must find the nullspace of

$$A - \lambda_2 I = A + 2I = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 9 \\ 0 & -1 & -3 \end{pmatrix}.$$

We see that the nullspace is generated by the eigenvector $\mathbf{v}_2 = (0, -3, 1)^T$, so it has dimension 1. Thus the geometric multiplicity of λ_2 is 1.

c) (3 points) For each eigenvalue, find a basis of the eigenspace.

Answer: In part b) we found that the eigenspace for λ_2 was generated by the eigenvector \mathbf{v}_2 . For $\lambda_1 = -1$ we examine the nullspace of

$$A - \lambda_1 I = A + I = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 9 \\ 0 & -1 & -4 \end{pmatrix}.$$

This matrix is almost in row echelon form, and we see easily that the nullspace is generated by the eigenvector $\mathbf{v}_1 = (1, -4, 1)^T$.

d) (3 points) Find a fundamental set of solutions for the system.

Answer: We must find one solution corresponding to the eigenvalue λ_1 , and two corresponding to λ_2 . For λ_1 we can use the eigenvector \mathbf{v}_1 , and we get the solution

$$\mathbf{y}_1(t) = e^{tA}\mathbf{v}_1 = e^{\lambda_1 t}\mathbf{v}_1 = e^{-t} \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}.$$

for λ_2 we get one solution from the eigenvector \mathbf{v}_2 ,

$$\mathbf{y}_2(t) = e^{tA}\mathbf{v}_2 = e^{\lambda_2 t}\mathbf{v}_2 = e^{-t} \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}.$$

For the second solution we must look for a generalized eigenvector. This will be a vector in the nullspace of

$$(A - \lambda_2 I)^2 = (A + 2I)^2 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

The nullspace has the basis $(0, 1, 0)^T$ and $(0, 0, 1)^T$. We need only one of these which is linearly independent from \mathbf{v}_2 . Let's choose $\mathbf{v}_3 = (0, 0, 1)^T$. Since $(A - \lambda_2 I)^2\mathbf{v}_3 = \mathbf{0}$, the solution is

$$\begin{aligned} \mathbf{y}_3(t) &= e^{tA}\mathbf{v}_3 \\ &= e^{\lambda_2 t}[\mathbf{v}_3 + t(A - \lambda_2 I)\mathbf{v}_3] \\ &= e^{-2t} \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 9 \\ 0 & -1 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right] \\ &= e^{-2t} \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 9 \\ -3 \end{pmatrix} \right] \\ &= e^{-2t} \begin{pmatrix} 0 \\ 9t \\ 1 - 3t \end{pmatrix}. \end{aligned}$$

e) (3 points) Find the solution which satisfies the initial condition $\mathbf{y}(0) = (1, 2, 0)^T$.

Answer: We need to find the constants C_1 , C_2 , and C_3 such that $\mathbf{y} = C_1\mathbf{y}_1 + C_2\mathbf{y}_2 + C_3\mathbf{y}_3$ satisfies the initial condition. This means $(1, 2, 0)^T = \mathbf{y}(0) = C_1\mathbf{y}_1(0) + C_2\mathbf{y}_2(0) + C_3\mathbf{y}_3(0) = C_1\mathbf{v}_1 + C_2\mathbf{v}_2 + C_3\mathbf{v}_3$, or

$$\begin{aligned} C_1 &= 1 \\ -4C_1 - 3C_2 &= 2 \\ C_1 + C_2 + C_3 &= 0. \end{aligned}$$

We easily find that $C_1 = 1$, $C_2 = -2$, and $C_3 = 1$. Hence the solution is

$$\mathbf{y}(t) = \mathbf{y}_1(t) - 2\mathbf{y}_2(t) + \mathbf{y}_3(t).$$

f) (3 points) What can you say about the type of the equilibrium point at the origin $(0, 0, 0)^T$?

Answer: Since all of the eigenvalues are negative, the origin is asymptotically stable.