Math 211

Final Exam

December 8, 1999

Part 2

Instructions: Write out and sign the honor pledge on your exam paper for Part 2. In addition put the name of your instructor in a prominent place on your exam. It is due by 3:30 PM on Monday, December 20, in the Mathematics Department Office, HB 220.

Part 1 is worth 60 points, and Part 2 is worth 40 points. Part 2 of the exam is an open book, open notes, take home exam. Because of academic regulations **there is a two hour time limit on this exam.** It is up to you to keep to this limit, but you should only count time actually spent working on the exam. Time spent dealing with computer problems does not count toward the two hour limit. You are allowed to take breaks, which also do not count toward the two hour limit, however, once you have started the exam it would be unfair to consult ODE material during your breaks. You may not discuss the exam with your fellow students. If you have any questions, consult one of the instructors for the course.

Please give reasons for all of your answers. Remember that some reasons are better than others. For example, it is better to refer to a theorem stated in class and/or in the books than to say "the computer printout shows that ...". Of course sometimes the computer printout is all you have.

In answering these questions you are not limited to the use of MATLAB, although it will be useful. You should use any combination of the analytic, qualitative, or numerical methods you have learned.

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1. The autonomous system

$$\frac{dx}{dt} = 6x\left(1 - \frac{x}{6}\right) - xy,$$
$$\frac{dy}{dt} = 10y\left(1 - \frac{y}{10}\right) - 2xy.$$

is a nonlinear model for two competing species with populations x and y.

a) (7 points) By hand, compute the nullclines and the equilibrium points for the system.

b) (7 points) Enter the system into `pplane5`. So far we have classified equilibrium points for *linear* systems as nodal sinks, nodal sources, saddle points, centers, spiral sinks, and spiral sources, but we can do the same for nonlinear systems like this one. If you zoom in far enough around any of the equilibrium points of the competing species system, the picture will begin to look like a nodal sink, nodal source, saddle point, center, spiral sink, or a spiral source. Use the "zoom in" feature in `pplane5` to zoom in around each of the equilibrium points for this system, and classify it as one of the six mentioned types. Print out and turn in the plot of the phase plane that you use to classify each of the equilibrium points. Which of the equilibrium points are stable? Which are unstable?

c) (6 points) Any sink has a *basin of attraction*. This is the set of points (x_0, y_0) with the property that the solution curve with initial values (x_0, y_0) approaches the sink as t gets large. For each sink in our competing species example, on the basis of the results of several solution curves, draw the basin of attraction. This means that on a computer printout of the phase plane you should (by hand) sketch approximately the curves that separate the basins of attaction. In this part we are only interested in the quadrant where both populations are nonnegative.

2. Suppose we have two identical pendulums which are separated by a short distance. Suppose the bobs are attracted to each other by a force that satisfies an inverse square law. This phenomenon might be modeled by the system

$$\theta_1'' = -\sin(\theta_1) + \frac{0.01}{(1-\theta_1+\theta_2)^2},$$
$$\theta_2'' = -\sin(\theta_2) - \frac{0.01}{(1-\theta_1+\theta_2)^2}.$$

Here θ_1 and θ_2 are the angular displacements of the two pendulums.

a) (6 points) Find a first order system that is equivalent to the above system.

b) (6 points) Write a MATLAB function m-file describing this system suitable for use with `ode45`.

c) (6 points) Use `ode45` to solve the system over the time interval $(0, 500)$, with each of the following initial conditions. In each case plot the two angles θ_1 and θ_2 versus time on the same figure, and submit it as part of your test submission.

i) $\theta_1(0) = -0.1$, $\theta_1'(0) = 0$, $\theta_2(0) = -0.1$, and $\theta_2'(0) = 0$

ii) $\theta_1(0) = -0.1$, $\theta_1'(0) = 0$, $\theta_2(0) = 0.1$, and $\theta_2'(0) = 0$

iii) $\theta_1(0) = -0.1$, $\theta_1'(0) = 0$, $\theta_2(0) = 0$, and $\theta_2'(0) = 0$

d) (2 points) In your opinion, in what way is the motion in case i) most different from the other two?

To answer this question, you will need to examine the solutions in more detail. You can get a closer picture of the plot of θ_1 and θ_2 versus time on any subinterval by using the zoomin tool. (Click on the magnifying glass with a plus in it on the Tool Bar to activate it. Then drag a rectangle over the part that you want to look at. Use the zoomout tool to go in the other direction.) You can get a different view of the motion by using the MATLAB command `comet` to dynamically plot θ_1 versus θ_2 (this is called a *configuration space plot*). To learn how to use `comet`, use the MATLAB command `help comet`. You should not submit the comet plot with your test, however.