Math 211

Final Exam

December 2, 1998

Part 2

Instructions: Write out and sign the honor pledge on your exam paper for Part 2. In addition put the name of your instructor in a prominent place on your exam. It is due by 3:30 PM on Saturday, December 19, in the Mathematics Department Office, HB 220.

Part 1 is worth 120 points, and Part 2 is worth 80 points. Part 2 of the exam is an open book, open notes, take home exam. Because of academic regulations **there is a two hour time limit on this exam.** It is up to you to keep to this limit, but you should only count time actually spent working on the exam. Once you have started the exam it would be unfair to consult ODE material during whatever breaks you allow yourself. You may not discuss the exam with your fellow students. If you have any questions, consult one of the instructors for the course.

Please give reasons for all of your answers. Remember that some reasons are better than others. For example, it is better to refer to a theorem stated in class and/or in the books than to say "the computer printout shows that ...". Of course sometimes the computer printout is all you have.

In answering these questions you are not limited to the use of MATLAB, although it will be useful. You should use any combination of the analytic, qualitative, or numerical methods you have learned.

The exam deals with a model of a situation where there are three different species which we will refer to as rabbits, deer, and wolves. We will let $x(t)$ denote the population of rabbits, $y(t)$ that of the deer, and $z(t)$ the wolves. These are measured in units of thousands.

After a period of observation some population biologists decide that the following is a good model of what happens to the three populations:

$$
\begin{aligned}
x' &= (1 - x - y - z/10)x = f(x, y, z) \\
y' &= (4 - 2x - 7y - z)y = g(x, y, z) \\
z' &= (-1 + x + 2y)z = h(x, y, z)
\end{aligned}
$$

It is your job to analyze this system and to determine the long term behavior of the populations predicted by this model. You can do so by answering the following questions.

1. (10 points) Assume that there are no wolves, but there are rabbits and deer. Examine the resulting two dimensional system to predict what happens to the populations of rabbits and deer as $t \to \infty$. To do this problem you can use `pplane5` and all of its capabilities, including computing and plotting nullclines, equilibrium points, linearizations, etc. However, your last resource should be computed solutions. At far as possible prove your statements using the methods of qualitative analysis you have learned.

2. (10 points) Assume that there are no rabbits, but there are wolves and deer. Examine the resulting two dimensional system to predict what happens to the populations of wolves and deer as $t \to \infty$. Once more use `pplane5` as you did in the previous problem.

3. (10 points) Assume that there are no deer, but there are wolves and rabbits. Examine the resulting two dimensional system to predict what happens to the populations of wolves and rabbits as $t \to \infty$. Once more use `pplane5` as you did in the previous problem.

4. (10 points) It is not difficult to see that the full system has 6 equilibrium points, but it has only one with **all of the variables positive.** Find this equilibrium point. (The answer is $\mathbf{p} = (9/13, 2/13, 20/13)^T$, however it is not acceptable to simply plug these numbers into the equations.)

5. (10 points) Compute the Jacobian of the full system at the equilibrium point **p** found in the previous problem. Do this problem in two steps:

(a) (8 points) Provide a formula for the Jacobian in terms of the partial derivatives of f, g, and h. Leave the actual computation of the derivatives to the next part.

(b) (2 points) Finish the computation. The answer is

$$J = \begin{pmatrix} -9/13 & -9/13 & -9/130 \\ -4/13 & -14/13 & -2/13 \\ 20/13 & 40/13 & 0 \end{pmatrix}.$$

(These computations are simple, but quite lengthy, and part (b) is worth only 2 points. It is a good idea to leave this part of the test to the last.)

6. (10 points) Describe what happens as $t \to \infty$ to solutions that start close to **p**.

7. (10 points) Use `ode45` to compute solutions for a number of different choices of initial conditions with all initial populations positive. The choices are up to you, but choose them to be quite different and far from the equilibrium point **p**. Submit one figure for each of three different cases. Explain what happens to the populations as $t \to \infty$. In particular, compare what happens to the solutions as $t \to \infty$ with the equilibrium point **p** itself.

8. (10 points) It can be shown that all of the other equilibrium points are unstable. Using this fact, and the work that you have done in Problems 5, 6 and 7, what do you conjecture is the behavior of the populations as $t \to \infty$ starting from an arbitrary initial set of populations, all of which are positive. What part of your conjecture can you actually prove and what part is based on experimental evidence?