

Math 211

Final Exam—Answers

December 3, 2000

1. (8 points) Find the solution to the initial value problem

$$\frac{dy}{dt} + y \tan t = \cos t \cdot \sin t \quad \text{with} \quad y(0) = 0.$$

What is the interval of existence of the solution?

**Answer:** The equation is linear. An integrating factor is

$$u(t) = e^{\int \tan t \, dt} = e^{-\ln \cos t} = 1/\cos t.$$

We check that

$$\left[ \frac{1}{\cos t} y \right]' = \frac{1}{\cos t} [y' + y \tan t] = \frac{1}{\cos t} [\cos t \cdot \sin t] = \sin t.$$

Integrating we get

$$\frac{1}{\cos t} y(t) = -\cos t + C.$$

Hence the general solution is

$$y(t) = -\cos^2 t + C \cos t.$$

Alternately, the student could first solve the homogeneous equation  $y' + y \tan t = 0$  to get the solution  $y_h(t) = \cos t$ . Using the method of variation of parameters, the solution to the inhomogeneous equation is written as  $y(t) = v(t) \cos t$ . Then

$$y'(t) = -\sin t \cdot v(t) + v'(t) \cos t = -\tan t \cdot y(t) + v'(t) \cos t.$$

Hence we must have  $v' = \sin t$ , or  $v(t) = -\cos t + C$ , and

$$y(t) = -\cos^2 t + C \cos t.$$

For the initial condition we have  $0 = y(0) = -1 + C$ , so  $C = 1$ , and

$$y(t) = \cos t - \cos^2 t.$$

To find the interval of existence, we first notice that  $\cos t - \cos^2 t$  is defined for all  $t$ . However, one of the coefficients of the differential equation is  $\tan t$ , which is continuous only on the interval  $(-\pi/2, \pi/2)$ . Since the differential equation only makes sense over the interval  $(-\pi/2, \pi/2)$ , the interval of existence can be no bigger than  $(-\pi/2, \pi/2)$ .

Give up to 5 points for the general solution, 2 points for the initial condition, and 1 point for the interval of existence.

2. (8 points) Solve the initial value problem

$$x'' + 6x' + 13x = 0 \quad \text{with} \quad x(0) = 2, x'(0) = -2.$$

**Answer:** The characteristic polynomial is  $\lambda^2 + 6\lambda + 13 = 0$ . By the quadratic formula or by completing the square, the roots are found to be  $\lambda = -3 \pm 2i$ . Thus the general solution is

$$y(t) = e^{-3t} [A \cos 2t + B \sin 2t].$$

For the initial conditions we differentiate to get

$$y'(t) = e^{-3t} [(2B - 3A) \cos 2t - (2A + 3B) \sin 2t].$$

Then the initial conditions give us

$$2 = y(0) = A$$

$$-2 = y'(0) = 2B - 3A.$$

The solution to this system is  $A = B = 2$ , and the solution to the initial value problem is

$$y(t) = 2e^{-3t} [\cos 2t + \sin 2t].$$

Give 2 points for the characteristic polynomial, or its equivalent in terms of exponential solutions. Give 3 points for the form of the general solution (real or complex). Give another 3 points for the final answer.

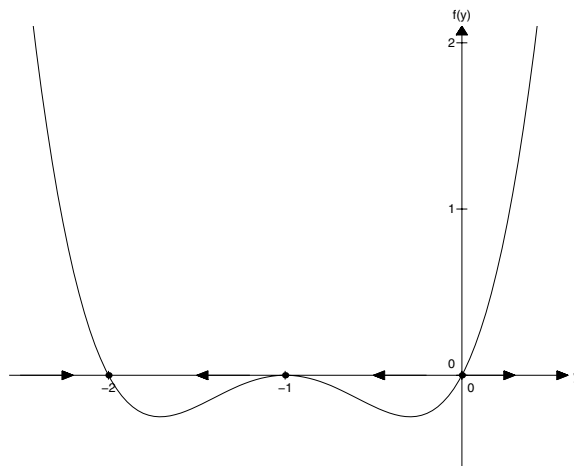
3. (8 points) Consider the equation

$$y' = (y^2 + 2y)(1 + y)^2.$$

a) Find and analyze all equilibrium points.

**Answer:** The equation can be written as  $y' = f(y)$ , where  $f(y) = y(y + 2)(y + 1)^2$ .

The equilibrium points are where  $f(y) = 0$ , or at  $y = -2$ ,  $-1$ , and  $0$ . The graph of  $f$  is shown in the following figure.

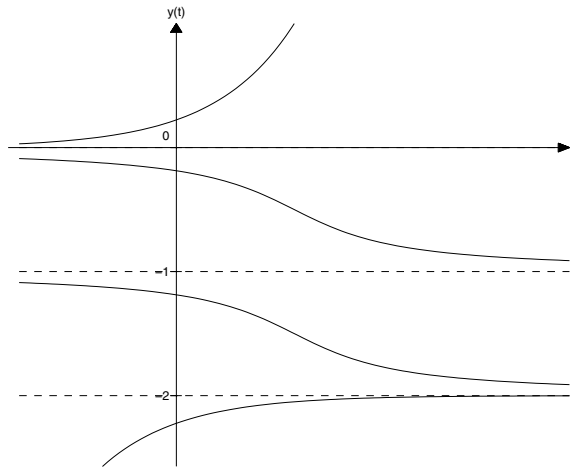


The graph of  $f(y) = y(y + 2)(y + 1)^2$ , showing the derived phase line information.

Also shown in the above figure is the phase line. It shows the equilibrium points and the direction of the solution flow in the intervals between them, as reflected in the sign of  $f(y)$ . From this information we readily conclude that  $-2$  is an asymptotically stable equilibrium point, and  $-1$  and  $0$  are unstable.

Part a) gets 4 points. 1 point for finding the equilibrium points correctly. 3 points for the analysis of the stability. Give full credit if they provide the graph of  $f$  and the phase line information. If they do not provide this, then they must provide algebraic information that is the equivalent of the phase line information.

b) Provide a sketch showing the qualitative behavior of all solutions. **Answer:**



Sketch of solutions in each of the four intervals determined by the equilibrium points.

Give 4 points for this sketch.

4. (10 points) Consider the system

$$x' = x(1 - y)$$

$$y' = x^2 - y.$$

a) Find and analyze the equilibrium points.

**Answer:** From the first equation we have  $x = 0$  or  $y = 1$ . From the second  $y = x^2$ . If  $x = 0$  we must have  $y = 0$ . If  $y = 1$ , we have  $x^2 = 1$ , or  $x = \pm 1$ . Thus there are three equilibrium points,  $(0, 0)$ ,  $(-1, 1)$ , and  $(1, 1)$ .

To analyze them we compute the Jacobian matrix

$$J = \begin{pmatrix} 1 - y & -x \\ 2x & -1 \end{pmatrix}.$$

At  $(0, 0)$

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since the matrix is diagonal, the entries on the diagonal are the eigenvalues. Hence  $(0, 0)$  is a saddle. At  $(-1, 1)$

$$J = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}.$$

Here the trace is  $T = -1$ , and the determinant is  $D = 2$ . We compute that  $T^2 - 4D = -7 < 0$ , so this is a spiral sink. At  $(1, 1)$

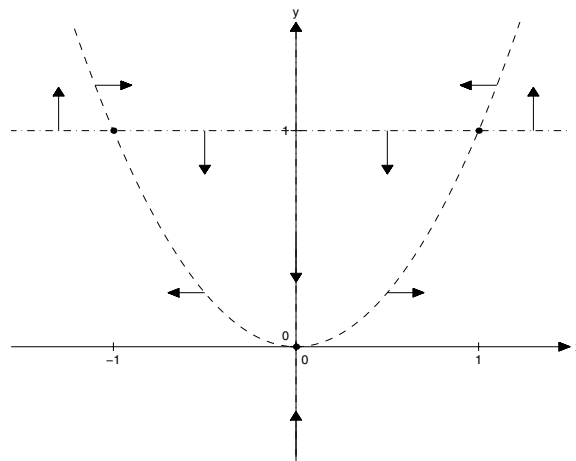
$$J = \begin{pmatrix} 0 & -1 \\ 2 & -1 \end{pmatrix}.$$

Again the trace is  $T = -1$ , the determinant is  $D = 2$ , and  $T^2 - 4D = -7 < 0$ , so this too is a spiral sink.

Give 6 points for part a).

- b) Sketch the nullclines. On your sketch indicate the direction of the vector field along the nullclines.

**Answer:** In the figure below, the  $x$ -nullcline is sketched with dot-dashes, and the  $y$ -nullcline with dashes. The  $y$ -axis is included in the  $x$ -nullcline. (The arrow at the top of the  $y$ -axis is not a field arrow.)



The nullclines and the direction of the vector field.

Give 4 points for this sketch.

5. (10 points) A spring with mass  $m = 2$ , damping constant  $\mu = 4$ , and spring constant  $k = 4$  is being driven by the force  $F(t) = 2 \cos 3t$ .

a) Show that the equation of motion can be written as

$$x'' + 2x' + 2x = \cos 3t.$$

**Answer:** (2 points) The differential equation for the vibrating spring is

$$mx'' + \mu x' + kx = F(t).$$

With the given parameters we get

$$2x'' + 4x' + 4x = 2 \cos 3t.$$

Dividing by the mass give the required equation.

b) Solve the initial value problem with  $x(0) = 1$  and  $x'(0) = 0$ .

**Answer:** (6 points) The characteristic equation is  $\lambda^2 + 2\lambda + 2 = 0$ . The roots are  $\lambda = -1 \pm i$ . Hence the general solution to the homogeneous equation is

$$x_h(t) = e^{-t} [A \cos t + B \sin t].$$

To find a particular solution to the inhomogeneous equation we use the complex method and look for a solution to  $z'' + 2z' + 2z = e^{3it}$  of the form  $z(t) = ae^{3it}$ . Then we will set  $x_p(t) = \text{real}(z(t))$ . Substituting into the equation we get

$$\begin{aligned} a[-9 + 6i + 2]e^{3it} &= e^{3it} \quad \text{or} \\ a &= \frac{1}{-7 + 6i} = \frac{-7 - 6i}{85}. \end{aligned}$$

Thus

$$\begin{aligned} z(t) &= \frac{-7 - 6i}{85} e^{3it} \\ &= \frac{-1}{85} (7 + 6i)(\cos 3t + i \sin 3t) \\ &= \frac{-1}{85} [(7 \cos 3t - 6 \sin 3t) + i(6 \cos 3t + 7 \sin 3t)] \end{aligned}$$

Hence

$$x_p(t) = \text{real}(z(t)) = \frac{-1}{85} (7 \cos 3t - 6 \sin 3t).$$

The general solution to the inhomogeneous equation is

$$\begin{aligned} x(t) &= x_p(t) + x_h(t) \\ &= \frac{-1}{85} (7 \cos 3t - 6 \sin 3t) + e^{-t} [A \cos t + B \sin t]. \end{aligned}$$

Differentiating we get

$$x'(t) = \frac{1}{85}(21 \sin 3t + 18 \cos 3t) + e^{-t} [(B - A) \cos t - (A + B) \sin t].$$

The initial conditions give us the equations

$$\begin{aligned}\frac{-7}{85} + A &= 1 \\ \frac{18}{85} + B - A &= 0.\end{aligned}$$

Solving, we find that  $A = 92/85$  and  $B = 74/85$ . Thus the solution to the initial value problem is

$$x(t) = \frac{1}{85} [6 \sin 3t - 7 \cos 3t + e^{-t}(92 \cos t + 74 \sin t)].$$

Give 2 points for the solution to the homogeneous equation, 2 points for the particular solution, and 2 points for the solution to the initial value problem.

c) What is the steady state behavior of the solution?

**Answer:** (2 points) The steady state behavior is the same as the particular solution,

$$x_p(t) = \frac{1}{85}(6 \sin 3t - 7 \cos 3t).$$

6. (8 points) Three populations exist together and interact in isolated circumstances. We will denote the populations by  $x_1(t)$ ,  $x_2(t)$ , and  $x_3(t)$ . They interact as follows:

- The first population preys upon the second and would die out if  $x_2 = 0$ .
- The second population preys upon the third. The second population would be able to survive if the third were not present, but its growth would be limited by the availability of resources.
- The third population would flourish in the absence of the other two, but its growth would be limited by the availability of resources.
- The first and third populations do not interact directly.

Model the interactions between the three populations with a system of differential equations. Please notice that you are not required to solve the equations.

**Answer:** The model is

$$\begin{aligned}x_1' &= x_1(-a_1 + b_1x_2) \\ x_2' &= x_2(a_2 - b_2x_2 - c_2x_1 + d_2x_3), \\ x_3' &= x_3(a_3 - b_3x_3 - c_3x_2)\end{aligned}$$

with all constants positive.

Take off one point for every missing constant, and one for every extraneous constant. Of course the total points must be between 0 and 8.

7. (10 points)

a) Suppose a differential equation has the form

$$\frac{dy}{dt} = f(y/t),$$

for some function  $f$ . Show that the substitution  $v = y/t$  (or  $y = tv$ ) transforms the equation into

$$t \frac{dv}{dt} + v = f(v),$$

which is a separable equation.

**Answer:** (5 points) Differentiating the equation  $y = tv$  we get

$$\frac{dy}{dt} = t \frac{dv}{dt} + v.$$

Hence from the differential equation we have

$$t \frac{dv}{dt} + v = \frac{dy}{dt} = f(y/t) = f(v).$$

b) Find the general solution to the equation

$$\frac{dy}{dt} = 2 \left( \frac{y}{t} \right) + \left( \frac{y}{t} \right)^2$$

for  $y(t)$ .

**Answer:** (5 points) Making the substitution suggested in part a), the equation becomes

$$t \frac{dv}{dt} + v = 2v + v^2.$$

Separating variables we get

$$\frac{dv}{v + v^2} = \frac{dt}{t}.$$

When we integrate using partial fractions or the integral table we get

$$v(t) = \frac{At}{1 - At}.$$

Finally we get

$$y(t) = tv(t) = \frac{At^2}{1 - At}.$$

Give 2 points for getting the correct separable equation, 2 points for the integration, and one point for finding  $y$  after getting  $v$ .

8. (8 points) Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

- a) Find two linearly independent eigenvectors associated with the eigenvalue  $\lambda = 0$  and one nonzero eigenvector associated with the eigenvalue  $\lambda = 3$ .

**Answer:** (5 points) For  $\lambda = 0$  we are looking for the null space of  $A$ . It can be seen directly from  $A$ , or after row operations to reduce  $A$  to row echelon form that

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

form a basis for the nullspace. However, there are many possible correct answers.

For  $\lambda = 3$  we have

$$A - 3I = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

Row operations can be used to reduce this to

$$\begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

is a vector in the nullspace, and therefore an eigenvector.

Give 3 points for the 2 basis vectors for  $\lambda = 0$  and 2 for  $\lambda = 3$ .

- b) Find a fundamental set of solutions to the system  $\mathbf{y}' = A\mathbf{y}$ .

**Answer:** (3 points) In part a) we computed a basis of eigenvectors. Hence we have the fundamental system of solutions

$$\mathbf{x}_1(t) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2(t) = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{x}_3(t) = e^{3t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

9. (10 points) Let

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

a) Show that

$$e^{tA} = e^{\lambda t} \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

**Answer:** Using  $tA = t\lambda I + t(A - \lambda I)$ , and the fact that the summands commute, we have

$$e^{tA} = e^{t\lambda I} e^{t(A-\lambda I)}.$$

\*

We compute that

$$A - \lambda I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$(A - \lambda I)^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$(A - \lambda I)^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence the series for  $e^{t(A-\lambda I)}$  truncates and we have

$$\begin{aligned} e^{t(A-\lambda I)} &= I + t(A - \lambda I) + \frac{t^2}{2}(A - \lambda I)^2 \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + t \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

The result follows from (\*)

Give 6 points for this part. Students should not be required to give as much detail as is here because they have a theorem about matrices with a single eigenvalue that they might quote.

b) Find the general solution to the system  $\mathbf{y}' = A\mathbf{y}$ .

**Answer:** (4 points) there are several ways to respond to this question.

For one thing, the solution to the initial value problem  $\mathbf{y}' = A\mathbf{y}$  with  $\mathbf{y}(0) = \mathbf{v}$  is  $\mathbf{y}(t) = e^{tA}\mathbf{v}$ .

They could also find generalized eigenvectors and write out a general solution.

10. (10 points) Consider the system

$$\begin{aligned}x' &= -1 - y + x^2 \\y' &= x + xy.\end{aligned}$$

a) Let  $r = \sqrt{x^2 + y^2}$ . Using the equation

$$r \frac{dr}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt},$$

show that any solution that starts on the circle  $r = 1$  must remain there for all  $t$ .

**Answer:** (5 points) Following the hint, we have

$$\begin{aligned}r \frac{dr}{dt} &= x \frac{dx}{dt} + y \frac{dy}{dt} \\&= x(-1 - y + x^2) + y(x + xy) \\&= x(-1 + x^2 + y^2) \\&= x(r^2 - 1).\end{aligned}$$

Hence if  $r = 1$  we have  $r' = 0$ , thus  $r$  is identically equal to 1, and the result follows.

b) Show that the disk  $\{(x, y) | x^2 + y^2 < 1\}$  is invariant.

**Answer:** (5 points) The disk is invariant by the uniqueness theorem. A solution starting in the disk cannot cross the boundary, since, by part a) any solution curve that hits the boundary circle lies entirely in the boundary

11. (10 points) The motion of a damped pendulum is described by the nonlinear second-order equation

$$\theta'' + \frac{c}{m}\theta' + \frac{g}{L}\sin\theta = 0,$$

where  $\theta$  is the angular displacement of the pendulum arm,  $L$  is the length of the pendulum arm,  $g = 9.8 \text{ m/s}^2$  is the acceleration due to gravity,  $c$  is a damping constant, and  $m$  is the mass of the pendulum bob.

a) Rewrite the second-order equation as a system of first-order equations.

**Answer:** (4 points) Introduce a new variable  $v = \theta'$ . the the system is

$$\begin{aligned}\theta' &= v \\v' &= -\frac{c}{m}v - \frac{g}{l}\sin\theta\end{aligned}$$

- b) Suppose that  $m = c = 1$ , and  $L = 4$ . Find all equilibrium points and classify them by type and stability.

**Answer:** (6 points) The equilibrium points are  $(k\pi, 0)$ , where  $k$  is an integer. The Jacobian is

$$J(\theta, v) = \begin{pmatrix} 0 & 1 \\ (-9.8/4) \cos \theta & -1 \end{pmatrix}.$$

When  $\theta = 2k\pi$ ,

$$J(2k\pi, 0) = \begin{pmatrix} 0 & 1 \\ -9.8/4 & -1 \end{pmatrix}.$$

We have  $T = -1$ ,  $D = 9.8/4$ , and  $T^2 - 4D = 1 - 9.8 < 0$ . Hence these are spiral sinks, and asymptotically stable.

When  $\theta = (2k + 1)\pi$ ,

$$J((2k + 1)\pi, 0) = \begin{pmatrix} 0 & 1 \\ 9.8/4 & -1 \end{pmatrix}.$$

We have  $D = -9.8/4 < 0$ , so these are saddle points.

Give 3 points for finding the equilibrium points, and 3 for the analysis.

### Table of Integrals

The letters  $a$ ,  $b$ ,  $c$ , and  $d$  stand for arbitrary constants.

$$\begin{aligned} \int \sec ax \, dx &= \frac{1}{a} \ln \tan \left( \frac{\pi}{4} + \frac{ax}{2} \right) \\ \int \tan ax \, dx &= -\frac{1}{a} \ln \cos ax \\ \int \cot ax \, dx &= \frac{1}{a} \ln \sin ax \\ \int \frac{ay + b}{cy + d} \, dy &= \frac{ay}{c} + \frac{bc - ad}{c^2} \ln(cy + d) \\ \int \frac{y}{(ay + b)(cy + d)} \, dy &= \frac{1}{bc - ad} \left[ \frac{b}{a} \ln(ay + b) - \frac{d}{c} \ln(cy + d) \right] \\ \int \frac{dy}{(ay + b)(cy + d)} &= \frac{1}{bc - ad} \ln \left( \frac{cy + d}{ay + b} \right) \end{aligned}$$