Math 211

Final Exam

December 3, 2000

Instructions: This is a closed book, three hour exam. Calculators are not allowed. Please give reasons for all of your answers.

There is a short table of integrals on the last page which might be of help to you.

Print your name:

Print your section number:

Write out and sign the pledge:

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1. (8 points) Find the solution to the initial value problem

$$\frac{dy}{dt} + y \tan t = \cos t \cdot \sin t \quad \text{with} \quad y(0) = 0.$$

What is the interval of existence of the solution?

2. (8 points) Solve the initial value problem

$$x'' + 6x' + 13x = 0 \quad \text{with} \quad x(0) = 2, x'(0) = -2.$$

3. (8 points) Consider the equation

$$y' = (y^2 + 2y)(1 + y)^2.$$

a) Find and analyze all equilibrium points.

b) Provide a sketch showing the qualitative behavior of all solutions.

4. (10 points) Consider the system

$$x' = x(1-y)$$
$$y' = x^2 - y.$$

a) Find and analyze the equilibrium points.

b) Sketch the nullclines. On your sketch indicate the direction of the vector field along the nullclines.

5. (10 points) A spring with mass $m = 2$, damping constant $\mu = 4$, and spring constant $k = 4$ is being driven by the force $F(t) = 2\cos 3t$.

a) Show that the equation of motion can be written as

$$x'' + 2x' + 2x = \cos 3t.$$

b) Solve the initial value problem with $x(0) = 1$ and $x'(0) = 0$.

c) What is the steady state behavior of the solution?

6. (8 points) Three populations exist together and interact in isolated circumstances. We will denote the populations by $x_1(t)$, $x_2(t)$, and $x_3(t)$. They interact as follows:

- The first population preys upon the second and would die out if $x_2 = 0$.
- The second population preys upon the third. The second population would be able to survive if the third were not present, but its growth would be limited by the availability of resources.
- The third population would flourish in the absence of the other two, but its growth would be limited by the availability of resources.
- The first and third populations do not interact directly.

Model the interactions between the three populations with a system of differential equations. Please notice that you are not required to solve the equations.

7. (10 points)

a) Suppose a differential equation has the form

$$\frac{dy}{dt} = f(y/t),$$

for some function f. Show that the substitution $v = y/t$ (or $y = tv$) transforms the equation into

$$t\frac{dv}{dt} + v = f(v),$$

which is a separable equation.

b) Find the general solution to the equation

$$\frac{dy}{dt} = 2\left(\frac{y}{t}\right) + \left(\frac{y}{t}\right)^2$$

for $y(t)$.

8. (8 points) Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

a) Find two linearly independent eigenvectors associated with the eigenvalue $\lambda = 0$ and one nonzero eigenvector associated with the eigenvalue $\lambda = 3$.

b) Find a fundamental set of solutions to the system $\mathbf{y}' = A\mathbf{y}$.

9. (10 points) Let

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

a) Show that

$$e^{tA} = e^{\lambda t}\begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

b) Find the general solution to the system $\mathbf{y}' = A\mathbf{y}$.

10. (10 points) Consider the system

$$x' = -1 - y + x^2$$
$$y' = x + xy.$$

a) Let $r = \sqrt{x^2 + y^2}$. Using the equation

$$r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt},$$

show that any solution that starts on the circle $r = 1$ must remain there for all t.

b) Show that the disk $\{(x, y) | x^2 + y^2 < 1\}$ is invariant.

11. (10 points) The motion of a damped pendulum is described by the nonlinear second-order equation

$$\theta'' + \frac{c}{m}\theta' + \frac{g}{L}\sin\theta = 0,$$

where θ is the angular displacement of the pendulum arm, L is the length of the pendulum arm, $g = 9.8\,\text{m/s}^2$ is the acceleration due to gravity, c is a damping constant, and m is the mass of the pendulum bob.

a) Rewrite the second-order equation as a system of first-order equations.

b) Suppose that $m = c = 1$, and $L = 4$. Find all equilibrium points and classify them by type and stability.

Table of Integrals

The letters a, b, c, and d stand for arbitrary constants.

$$\int \sec ax\,dx = \frac{1}{a}\ln\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

$$\int \tan ax\,dx = -\frac{1}{a}\ln\cos ax$$

$$\int \cot ax\,dx = \frac{1}{a}\ln\sin ax$$

$$\int \frac{ay+b}{cy+d}\,dy = \frac{ay}{c} + \frac{bc-ad}{c^2}\ln(cy+d)$$

$$\int \frac{y}{(ay+b)(cy+d)}\,dy = \frac{1}{bc-ad}\left[\frac{b}{a}\ln(ay+b) - \frac{d}{c}\ln(cy+d)\right]$$

$$\int \frac{dy}{(ay+b)(cy+d)} = \frac{1}{bc-ad}\ln\left(\frac{cy+d}{ay+b}\right)$$