Midterm 1 Solutions: Math 211

1. We have the linear problem $\dfrac{dy}{dt} + \dfrac{y}{t} = 3t + 2, \quad y(-1) = 2$

   The integrating factor is $e^{\int \frac{1}{t}\,dt} = e^{\ln t} = t$. Hence

$$\begin{aligned} \frac{d}{dt}(ty) &= 3t^2 + 2t \\ \int \frac{d}{dt}(ty)\,dt &= \int (3t^2 + 2t)\,dt \\ ty &= t^3 + t^2 + C \\ y &= t^2 + t + C/t \end{aligned}$$

   The initial condition $y(-1) = 2$ gives

$$\begin{aligned} &2 = 1 - 1 - C, \qquad \text{so } \; C = -2. \\ &y = t^2 + t - 2/t. \end{aligned}$$

   Note: It is **not** correct to write $\{t \in \mathbb{R} : t \neq 0\}$ as the interval. They must identify that, since $t_0 = -1$, the interval $(-\infty, 0)$ is the correct choice.

2. We have the separable problem $\dfrac{dy}{dt} = \dfrac{2t}{y + yt^2}, \quad y(0) = 2.$

   Separating variables, we arrive at $y\,dy = \dfrac{2t}{1+t^2}\,dt$.

   We integrate $\displaystyle\int y\,dy = \int \frac{2t}{1+t^2}\,dt$ to get

$$y^2 = \ln(1 + t^2) + C'.$$

   So $y^2 = 2\ln(1 + t^2) + C$. Since $y(0) = 2$, we have

$$4 = 2\ln(1) + C. \quad \text{So } \; C = 4.$$

   This gives $y^2 = 2\ln(1+t^2)+4$; equivalently we have $y = \pm\sqrt{2\ln(1 + t^2) + 4}$.

   Note: Students must acknowledge that there is a $\pm$, but that the positive is chosen because $y_0 > 0$. Deduct points for solutions that do not make this reasoning.

   Hence our solution is $y = \sqrt{2\ln(1 + t^2) + 4}$.

3. The equilibrium points are at $x = a$, $x = b$.

(a)

Hence $x = a$ is an asymptotically stable solution and $x = b$ is unstable.

(b) The graph (which is not necessary) is

For **all** values of $x(0)$ strictly between 0 and b, the solutions $x(t)$ tend to a.

4. If $S(t)$ is the amount of salt in the tank at time t, then $\frac{dS}{dt}$ = rate in − rate out. Clearly one can write

$$\frac{dS}{dt} = 2 + 4 - \frac{S}{100 + t}(3),$$

or

$$\frac{dS}{dt} + \frac{3S}{100 + t} = 6.$$

This is a linear equation with integrating factor

$$\mu(t) = e^{\int \frac{3}{100+t}\,dt} = e^{3\ln(100+t)} = (100 + t)^3.$$

Multiplying by the integrating factor, we have

$$\frac{d}{dt}((100+t)^3 S(t)) = 6(100+t)^3,$$

or

$$(100+t)^3 S(t) = \frac{3}{2}(100+t)^4 + C$$

or

$$S(t) = \frac{3}{2}(100+t) + C/(100+t)^3.$$

Since $S(0) = 0$, we have

$$0 = \frac{3}{2} \cdot 100 + \frac{C}{100^3},$$

or $C = -150 \cdot (100)^3$. Hence the concentration equals

$$\begin{aligned}
\frac{S(100)}{100+100} &= \frac{1}{200}\left[\frac{3}{2} \cdot 200 + \frac{-150 \cdot 100^3}{200^3}\right] \\
&= \frac{1}{200}\left[300 - \frac{150}{8}\right] \\
&= \frac{1}{20}\left[30 - \frac{15}{8}\right] = \frac{1}{20}\left(\frac{240-15}{8}\right) \\
&= \frac{225}{160} = \frac{45}{32}.
\end{aligned}$$

5. (a) The differential equation is $\frac{dP}{dt} = P - kt$. Students may also write $P(0) = P_0$, although this is not required.

(b) We have $P(2) = 60$ and $\frac{dP}{dt} = 20$ at $t = 2$.

So $20 = 60 - k(2)$. Hence $k = 20$. The equation $\frac{dP}{dt} = P - 20t$, or equivalently $\frac{dP}{dt} - P = -20t$, can be solved by multiplying through by the integrating factor e^{-t}. We then have $\frac{d}{dt}e^{-t}P(t) = -20te^{-t}$, which integrates to $e^{-t}P(t) = 20te^{-t} + 20e^{-t} + C$. Hence $P(t) = 20t + 20 + ce^t$.
When $t = 2$, $P = 60$, so

$$\begin{aligned}
60 &= 20(2) + 20 + ce^2 \\
60 &= 60 + ce^2, \quad \text{so } c = 0. \\
P(t) &= 20t + 20
\end{aligned}$$

The initial population is $P(0) = 20$.

6. We have $m\frac{dv}{dt} = -mg - kv(1 + a|v|)$ where $v < 0$, $k > 0$ and $g > 0$.

We can first remove the absolute value sign by replacing $|v|$ with $-v$.

$$\begin{aligned} m\frac{dv}{dt} &= -mg - kv(1 - av) \\ &= -mg - kv + akv^2. \end{aligned}$$

This is an autonomous system and so we can find equilibrium points by setting the right side to zero. By the quadratic equation,

$$\begin{aligned} akv^2 - kv - mg = 0, \\ v_{eq} = \frac{k \pm \sqrt{k^2 + 4mgak}}{2ak}. \end{aligned}$$

Since the velocity < 0 all at all $t > 0$, we must choose the negative solution, so the terminal velocity is

$$v = \frac{k - \sqrt{k^2 + 4mgak}}{2ak}.$$

7. (a)

$$\text{RHS} \; = \frac{t^2 - 4(t-2) - 3}{1 + (t-2)^2} = \frac{t^2 - 4t + 8 - 3}{1 + t^2 - 4t + 4} = \frac{t^2 - 4t + 5}{t^2 - 4t + 5} = 1$$

In addition, LHS $= \frac{d}{dt}(t - 2) = 1$ Since these are identical for all t, the function $t - 2$ is a solution.

Note: Students should **not** equate the right and left side until they know they are equal.

(b) Note that $f(t, y) = \frac{t^2 - 4y - 3}{1 + y^2}$ is continuous and that

$$\frac{\partial f}{\partial y} = \frac{-4(1 + y^2) - 2y(t^2 - 4y - 3)}{(1 + y^2)^2}$$

exists and is continuous everywhere, so there exists a solution $\widetilde{y}(t)$ passing through $(0, 0)$, and this solution $\widetilde{y}(t)$ cannot intersect $y = t-2$ by uniqueness. Since $(0, 0)$ is above $y = t - 2$, the solution $\widetilde{y}(t)$ must tend to ∞ as $t \to \infty$.