

Mathematics 211 Midterm 2, November 14, 2000

1. For the following matrices  $\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 4 \\ 2 & 0 & 5 \end{pmatrix}$  and  $\mathbf{B} = \begin{pmatrix} 3 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & 1 & 2 \end{pmatrix}$  verify the following identities (i.e. calculate LHS and RHS and show they agree).
- (a)  $\det(\mathbf{A}\mathbf{B}) = \det(\mathbf{A})\det(\mathbf{B})$ .
- (b)  $\text{tr}(\mathbf{A} + \mathbf{B}) = \text{tr}(\mathbf{A}) + \text{tr}(\mathbf{B})$ .
2. (a) Find a fundamental set of solutions of the form  $e^{\lambda t}\mathbf{v}$  for the system  $\mathbf{y}' = \mathbf{A}\mathbf{y}$ , where

$$\mathbf{A} = \begin{pmatrix} -1 & 2 \\ -1 & -4 \end{pmatrix}$$

- (b) Verify that the functions you produced in part (a) are indeed solutions i.e. that they satisfy the system.
- (c) Verify directly that the solutions you produced from part (a) are linearly independent solutions.
- (d) Find the solution satisfying the initial condition  $\mathbf{y}(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ .
3. (a) Suppose that  $\mathbf{A}^2 = \alpha\mathbf{A}$ . Show that  $e^{t\mathbf{A}} = \mathbf{I} + \left(\frac{e^{\alpha t}-1}{\alpha}\right)\mathbf{A}$ .
- (b) Find a set of fundamental solutions to the system  $\mathbf{y}' = \mathbf{A}\mathbf{y}$  where  $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ . NOTE: part (a) may be useful.

4. Determine the type (i.e. saddle point, nodal source, etc.) of the equilibrium point at  $\mathbf{0}$  for the system  $\mathbf{y}' = \mathbf{A}\mathbf{y}$  in each of the following four cases:

(a)  $\mathbf{A} = \begin{pmatrix} -7 & 5 \\ -10 & 8 \end{pmatrix}$ .

(b)  $\mathbf{A} = \begin{pmatrix} 0 & 6 \\ -2 & 7 \end{pmatrix}$ .

(c)  $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$ .

(d)  $\mathbf{A} = \begin{pmatrix} -4 & -2 \\ 10 & 4 \end{pmatrix}$ .

5. (a) Write  $y'' + py' + y = 0$  as a linear system, where  $p$  is a real number.
- (b) What condition on  $p$  will give:
- (i) a nodal sink?
- (ii) a center?
- (iii) a spiral source?
- (c) If we have a center, what direction is the rotation (i.e. clockwise or counter-clockwise)? **NOTE:** to receive credit you must have supporting calculations/explanations for your answer.

6. (a) For what values of  $x$  does the matrix  $\begin{pmatrix} -1-x & 4 & 2 \\ 0 & -x & -2 \\ 0 & 8 & 8-x \end{pmatrix}$  have a non-trivial nullspace?
- (b) Find a fundamental set of solutions to the system  $\mathbf{y}' = \mathbf{A}\mathbf{y}$  where  $\mathbf{A} = \begin{pmatrix} -1 & 4 & 2 \\ 0 & 0 & -2 \\ 0 & 8 & 8 \end{pmatrix}$ .