Math 211

Final Exam

Spring 2003

You have **3 hours** to complete this test.

Make sure to show your work and justify your arguments.

Calculator policy: You may use calculators to evaluate standard functions on floating point numbers (like $\sqrt{3.12}$, $\ln(35/7)$, or $\sin(\pi/17)$). You may not use symbolic operations, numerical integration, or any graphing functions.

1. (10p) Solve the initial value problem

$$y' = \frac{t(t^2+1)}{4y^3}, \qquad y(0) = -1/\sqrt{2}.$$

Solution This is a separable equation, after separating the variables we obtain

$$4y^3y' = t(t^2+1),$$

and integration gives us

$$y^4 = \frac{(t^2+1)^2}{4} + C.$$

At this point we can find C by the initial value condition, we obtain $C = 0$ and the solution is

$$\boxed{y = -\sqrt{\frac{t^2+1}{2}}.}$$

2. (10p) Solve the initial value problem

$$t^3y' + 4t^2y = e^{-t}, \qquad y(-1) = 0.$$

Solution We divide by t^3 to put the equation into normal form then find the integrating factor $e^{\int 4/t dt} = t^4$ and the equation becomes

$$t^4y' + 4t^3y = te^{-t}.$$

This implies that

$$t^4y = -te^{-t} - e^{-t} + C$$

and again we can find C by the initial value condition to be $C = 0$. We obtain that

$$\boxed{y = -\frac{1+t}{t^4}\,e^{-t}.}$$

3. (15p) Consider the periodically forced harmonic ordinary differential equation

$$y'' + 2y' + 2y = 3\cos 2t.$$

(a) Find a fundamental set of solutions of the associated homogeneous equation.
(b) Find the general solution of the inhomogeneous equation.
(c) Find the steady-state solution and determine its amplitude.
Solution (a) The characteristic equation is $\lambda^2+2\lambda+2=0$, so the eigenvalues are $\lambda_{1,2} = -1 \pm i$. This means that a fundamental set of solutions is

$$\boxed{\{e^{-t}\cos t, e^{-t}\sin t\}.}$$

(b) Using the complex method we try to find a solution in the form $z = Ae^{2it}$, then check the real part. ($2i$ is not an eigenvalue, so we will find a particular solution for the equation in this form.) After differentiating twice, we obtain that

$$-4A + 4iA + 2A = 3,$$

which means $A = \frac{3}{-2+4i} = -\frac{3}{10} - \frac{3}{5}i$. So the particular solution we looked for is $-\frac{3}{10}\cos 2t + \frac{3}{5}\sin 2t$ and the general solution is

$$\boxed{y = C_1 e^{-t}\cos t + C_2 e^{-t}\sin t - \frac{3}{10}\cos 2t + \frac{3}{5}\sin 2t.}$$

(c) The steady-state solution is the previous particular solution, so the amplitude is

$$\boxed{\sqrt{\left(\frac{3}{10}\right)^2 + \left(\frac{3}{5}\right)^2} = \frac{3\sqrt{5}}{10}.}$$

4. (15p) Find a fundamental set of solutions for the following system:

$$\begin{aligned} x' &= \ 3x - y \\ y' &= \ x + y. \end{aligned}$$

Also, find the solution to the initial value problem if $x(0) = 1$ and $y(0) = 2$.
Solution The characteristic polynomial is $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$, so 2 is a double eigenvalue. The nullspace of the matrix $A - 2I = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ is one dimensional with basis $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. This means that one solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The other solution can be found by any of the methods we discussed. Let's choose $w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $(A - 2I)w = v$ and this w is a generalized eigenvector, and the second solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{2t}(w + tv) = e^{2t} \begin{bmatrix} 1+t \\ t \end{bmatrix}.$$

These two are linearly independent, so a fundamental set of solutions is

$$\boxed{\{e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, e^{2t} \begin{bmatrix} 1+t \\ t \end{bmatrix}\}.}$$

The solution to the IVP can be found as a linear combination of these two solutions, say $az_1(t) + bz_2(t)$. The initial value conditions give us the system $a + b = 1$ and $a = 2$, so $b = -1$ and the solution of the IVP is

$$\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e^{2t} - te^{2t} \\ 2e^{2t} - te^{2t} \end{bmatrix}.}$$

5. (15p) Consider the system of ordinary differential equations

$$\begin{aligned} x' &= -4y + 2xy - 8 \\ y' &= 4y^2 - x^2. \end{aligned}$$

(a) Find all equilibrium points.
(b) Compute the Jacobian matrix.
(c) Classify the equilibrium points (as sinks, sources or saddles).

Solution (a) The second equation gives us that $2y = x$ or $2y = -x$. If $2y = x$, then the first equation becomes $x^2 - 2x - 8 = 0$, which has two solutions, $x = 4$ and $x = -2$. If $2y = -x$, then the first equation becomes $x^2 - 2x + 8 = 0$ which has no real solutions. Thus, we obtained two equilibrium points:

$$\boxed{(x, y) = (4, 2)} \text{ and } \boxed{(x, y) = (-2, -1).}$$

(b) The Jacobian matrix is

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 2y & -4 + 2x \\ -2x & 8y \end{bmatrix}.$$

(c) The Jacobian at the equilibrium point $(4, 2)$ is $\begin{bmatrix} 4 & 4 \\ -8 & 16 \end{bmatrix}$, so the eigenvalues are 12 and 8 and this is a (nodal) source. The Jacobian at the equilibrium point $(-2, -1)$ is $\begin{bmatrix} -2 & -8 \\ 4 & -8 \end{bmatrix}$, so the eigenvalues are $-5 + \sqrt{23}i$ and $-5 - \sqrt{23}i$ and this is a (spiral) sink.

6. (10p) Find the nullspace of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 4 & 3 & -4 \end{bmatrix}.$$

Solution The reduced row echelon form for this matrix is

$$A = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can see from here that the pivots are in column 2 and 3. The other two are free variables, so if we use the coordinates (x, y, x, v) then $x = t$ is free, $v = s$ is free and $z = 4s$ and $y = -2s$ from the first two rows. This means that the nullspace is

$$\boxed{\begin{bmatrix} x \\ y \\ z \\ v \end{bmatrix} = \begin{bmatrix} t \\ -2s \\ 4s \\ s \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -2 \\ 4 \\ 1 \end{bmatrix}.}$$

7. (10p) Consider the system of ordinary differential equations

$$\begin{aligned} x' &= -x \\ y' &= 2y + x^2. \end{aligned}$$

(a) Verify that the solution of the initial value problem where $x(0) = c$ and $y(0) = d$ is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} ce^{-t} \\ (d + \frac{c^2}{4})e^{2t} - \frac{c^2}{4}e^{-2t} \end{bmatrix}.$$

(b) Show that the set

$$S = \{\ (x, y) \in \mathbb{R}^2 \ : \ y = -\frac{x^2}{4}\ \}$$

is invariant.

Solution (a) Differentiate and plug in. Also, $x(0) = c$ and $y(0) = d$.
(b) If we start a solution on S, that means that $y(0) = -x^2(0)/4$ or $d = -c^2/4$. But then $x(t) = ce^{-t}$ and $y(t) = -c^2e^{-2t}/4$, so $y(t) = -x^2(t)/4$ for every t, so $(x(t), y(t)) \in S$ for every t and S is invariant.

8. (15p) Suppose that $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are three linearly independent vectors in $\mathbb{R}^3$. Prove that the vectors $\{\ \mathbf{x}+\mathbf{y},\ \mathbf{y}+\mathbf{z},\ \mathbf{z}+\mathbf{x}\ \}$ are also linearly independent.

Solution Suppose that

$$a(\mathbf{x}+\mathbf{y})+b(\mathbf{y}+\mathbf{z})+c(\mathbf{z}+\mathbf{x})=0.$$

(Where a, b, c are real constants.) This means that

$$(a+c)\mathbf{x}+(a+b)\mathbf{y}+(b+c)\mathbf{z}=0.$$

We know these vectors are linearly independent, so $a+c=0$, $a+b=0$ and $b+c=0$. But the only solution this system has for a, b, c is $a=0$, $b=0$ and $c=0$. (Check this.) So we obtained that the assumption

$$a(\mathbf{x}+\mathbf{y})+b(\mathbf{y}+\mathbf{z})+c(\mathbf{z}+\mathbf{x})=0$$

implies that $a=b=c=0$. That is the definition of linear independence and we are done.

9. (15p) Consider the system of ordinary differential equations

$$\begin{aligned} x' &= -ax+y+ay^2 \\ y' &= (1-a)x+xy. \end{aligned}$$

(a is a real constant.) Determine the stability of the equilibrium solution $(0,0)$ for every $a \in \mathbb{R}$.
Solution The Jacobian in this case is

$$J=\begin{bmatrix} -a & 1+2ay \\ 1-a+y & x \end{bmatrix}.$$

At the origin, this matrix is

$$J=\begin{bmatrix} -a & 1 \\ 1-a & 0 \end{bmatrix}.$$

So the eigenvalues are the roots of the characteristic equation,

$$\det(J-\lambda I)=\det\begin{bmatrix} -a-\lambda & 1 \\ 1-a & -\lambda \end{bmatrix}=\lambda^2+a\lambda+(a-1)=0.$$

The roots of this are

$$\lambda_{1,2}=\frac{-a\pm\sqrt{a^2-4(a-1)}}{2}=\frac{-a\pm\sqrt{(a-2)^2}}{2}=\frac{-a\pm(a-2)}{2}.$$

So we get two eigenvalues, $\lambda=-1$ and $\lambda=1-a$. By our linearization theorem if $a>1$, the origin is asymptotically stable, if $a<1$, the origin is unstable. Now if $a = 1$, the linearization theorem does not work. But in this case the system of ODE's is just

$$\begin{aligned} x' &= -x + y + y^2 \\ y' &= xy. \end{aligned}$$

We can find a solution of this fairly easily if we observe that if $x = y$, then both equations are just $x' = x^2$. But that has the solution $x(t) = -\frac{1}{t+C}$, so if we suppose $x(0) = y(0) = x_0 > 0$, then the solution is $x(t) = y(t) = \frac{x_0}{1-tx_0}$, which goes to infinity as t approaches $1/x_0$. So the origin is unstable if $a = 1$. (This last part is only for demonstration purposes; if you stated that at $a = 1$ we do not have a result from linearization, you got maximum credit.)