

Math 211
Final Exam
Spring 2003

You have **3 hours** to complete this test.

Make sure to show your work and justify your arguments.

Calculator policy: You may use calculators to evaluate standard functions on floating point numbers (like $\sqrt{3.12}$, $\ln(35/7)$, or $\sin(\pi/17)$). You may not use symbolic operations, numerical integration, or any graphing functions.

1. (10p) Solve the initial value problem

$$y' = \frac{t(t^2 + 1)}{4y^3}, \quad y(0) = -1/\sqrt{2}.$$

2. (10p) Solve the initial value problem

$$t^3 y' + 4t^2 y = e^{-t}, \quad y(-1) = 0.$$

3. (15p) Consider the periodically forced harmonic ordinary differential equation

$$y'' + 2y' + 2y = 3 \cos 2t.$$

(a) Find a fundamental set of solutions of the associated homogeneous equation.

(b) Find the general solution of the inhomogeneous equation.

(c) Find the steady-state solution and determine its amplitude.

4. (15p) Find a fundamental set of solutions for the following system:

$$\begin{aligned}x' &= 3x - y \\y' &= x + y.\end{aligned}$$

Also, find the solution to the initial value problem if $x(0) = 1$ and $y(0) = 2$.

5. (15p) Consider the system of ordinary differential equations

$$\begin{aligned}x' &= -4y + 2xy - 8 \\y' &= 4y^2 - x^2.\end{aligned}$$

(a) Find all equilibrium points.

(b) Compute the Jacobian matrix.

(c) Classify the equilibrium points (as sinks, sources or saddles).

EXAM CONTINUES ON BACK PAGE

6. (10p) Find the nullspace of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 4 & 3 & -4 \end{bmatrix}.$$

7. (10p) Consider the system of ordinary differential equations

$$\begin{aligned} x' &= -x \\ y' &= 2y + x^2. \end{aligned}$$

- (a) Verify that the solution of the initial value problem where $x(0) = c$ and $y(0) = d$ is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} ce^{-t} \\ (d + \frac{c^2}{4})e^{2t} - \frac{c^2}{4}e^{-2t} \end{bmatrix}.$$

- (b) Show that the set

$$S = \left\{ (x, y) \in \mathbb{R}^2 : y = -\frac{x^2}{4} \right\}$$

is invariant.

Solve only one of the following two problems:

8. (15p) Suppose that \mathbf{x} , \mathbf{y} and \mathbf{z} are three linearly independent vectors in \mathbb{R}^3 . Prove that the vectors $\{\mathbf{x} + \mathbf{y}, \mathbf{y} + \mathbf{z}, \mathbf{z} + \mathbf{x}\}$ are also linearly independent.
9. (15p) Consider the system of ordinary differential equations

$$\begin{aligned} x' &= -ax + y + ay^2 \\ y' &= (1 - a)x + xy. \end{aligned}$$

- (a is a real constant.) Determine the stability of the equilibrium solution $(0, 0)$ for every $a \in \mathbb{R}$.