Math 211
Final Solutions
December 2001

1) Consider the ordinary differential equation

$$yy' = 2t\sqrt{y^2+4}.$$

(a)(4p) Find the general solution.

Separate the variables and integrate:

$$\int \frac{y}{\sqrt{y^2+4}}dy = \int 2tdt,$$

so

$$\sqrt{y^2+4} = t^2 + C$$

and we obtain the general solution

$$\boxed{y^2 = (t^2+C)^2 - 4.}$$

(b)(4p) Find the solution of the initial value problem when $y(2) = -\sqrt{5}$.

Using the previous part we obtain $C = -1$ and then

$$\boxed{y = -\sqrt{(t^2-1)^2 - 4}.}$$

(c)(4p) Find the interval of existence of this solution.

We need $(t^2-1)^2 > 4$, i.e. $t^2 - 1 > 2$ or $t^2 > 3$ and the interval contains 2, so it is $\boxed{[\sqrt{3}, \infty).}$

2)(10p) Find the general solution of the ordinary differential equation.

$$t^2y' + 3ty = \frac{\sin t}{t} \qquad (t > 0).$$

The integrating factor is

$$e^{\int \frac{3}{t}dt} = e^{3\ln t} = e^{\ln t^3} = t^3,$$

the differential equation becomes

$$t^3y' + 3t^2y = \sin t,$$

so

$$(t^3y)' = \sin t$$

and then

$$t^3y = -\cos t + C$$

and we obtain the general solution

$$\boxed{y = \frac{C - \cos t}{t^3}.}$$

3)(10p) Three populations exist together and interact in isolated circumstances. Denote the populations by $x_1(t)$, $x_2(t)$ and $x_3(t)$. They interact as follows:

- The first population preys upon the other two and would decline dramatically if $x_2 = x_3 = 0$.
- The second population preys upon the third and would be able to survive if the third were not present, but its growth would be limited by the resources.
- The third population would flourish in the absence of the other two, but its growth would be limited by the resources.

Model the interactions between the three populations with a system of ordinary differential equations. You are **not** required to solve the equations.

One possible model is

$$\begin{aligned} x_1' &= \quad (-a + bx_2 + cx_3)x_1 \\ x_2' &= (d - ex_2 - fx_1 + gx_3)x_2 \\ x_3' &= (h - ix_3 - jx_1 - kx_2)x_3, \end{aligned}$$

where the constants $a - k$ are all positive.

4) Consider the system of ordinary differential equations

$$y' = Ay, \text{ where } A = \begin{bmatrix} -1 & 1 & 1 \\ -4 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a)(3p) Find a fundamental set of solutions.

First we find the eigenvalues and the corresponding eigenvectors. Evaluate the following determinant by the last row:

$$\det \begin{bmatrix} -1-\lambda & 1 & 1 \\ -4 & 3-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)((-1-\lambda)(3-\lambda)+4) =$$

$$= (1-\lambda)(\lambda^2 - 2\lambda + 1) = (1-\lambda)(\lambda - 1)^2.$$

So $\boxed{\lambda = 1 \text{ is a triple eigenvalue.}}$ Eigenvectors:

$$A - I = \begin{bmatrix} -2 & 1 & 1 \\ -4 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

and we see that the geometric multiplicity is 2, having two free variables. Two eigenvectors: $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$. So we have two solutions in the form

$$\boxed{\Phi_1(t) = e^t \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \qquad \Phi_2(t) = e^t \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.}$$

We can find a third by looking at the nullspace of $(A - I)^2 = 0$, so we obtain that a generalized eigenvector is $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and a solution is

$$\boxed{\Phi_3(t) = e^t[I + t(A - I)]v = e^t \begin{bmatrix} 1 - 2t \\ -4t \\ 0 \end{bmatrix}.}$$

(b)(3p) Find the general solution of the system.

The general solution is

$$\boxed{y(t) = c_1\Phi_1(t) + c_2\Phi_2(t) + c_3\Phi_3(t).}$$

(c)(3p) Find the solution of the initial value problem when $y(0) = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$.

We find from the previous part that $y(0) = \begin{bmatrix} c_1 + c_2 + c_3 \\ 2c_1 \\ 2c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$, so $c_1 = c_2 = c_3 = 1$ and the solution we look for is

$$\boxed{y(t) = \begin{bmatrix} 3e^t - 2te^t \\ 2e^t - 4te^t \\ 2e^t \end{bmatrix}.}$$

(d)(2p) What can we say about the stability of the equilibrium point at the origin?

The real parts of the eigenvalues are positive, so the equlibrium point is $\boxed{\text{unstable.}}$

2nd Solution. (a) (b) (c) We have one eigenvalue with multiplicity 3, so we can actually evaluate e^{At} easily. It is easy to check that $(A - I)^2 = 0$, so

$$e^{At} = e^t[I + (A - I)t] = e^t \begin{bmatrix} 1 - 2t & t & t \\ -4t & 1 + 2t & 2t \\ 0 & 0 & 1 \end{bmatrix}.$$

Then the answer to (a): the columns of this matrix. The answer to (b): $e^{At}c$ where $c \in \mathbb{R}^3$. The answer to (c): $e^{At}\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$.

5) Consider the periodically forced harmonic ordinary differential equation

$$y'' + 2y' + 2y = 4\cos 3t.$$

(a)(3p) Find a fundamental set of solutions of the associated homogeneous equation.

Characteristic equation: $\lambda^2 + 2\lambda + 2 = 0$, roots are $\lambda_1 = -1 + i$ and $\lambda_2 = \overline{\lambda_1} = -1 - i$. This means that a fundamental solution set is

$$\boxed{\Phi_1(t) = e^{-t}\cos t \qquad \Phi_2(t) = e^{-t}\sin t.}$$

(b)(3p) What is the steady-state solution?

Complex method: the inhomogeneneous term is the real part of $4e^{3it}$, so we look for a solution in the form $z(t) = Ae^{3it}$ and the real part of that will be our solution. By plugging $z(t)$ into the equation we obtain

$$(-7 + 6i)Ae^{3it} = 4e^{3it},$$

so $A = \frac{4}{-7+6i} = \frac{4(-7-6i)}{85} = -\frac{28}{85} - \frac{24}{85}i$. This gives us the solution

$$\boxed{y(t) = -\frac{28}{85}\cos 3t + \frac{24}{85}\sin 3t.}$$

Real method: try to find a solution in the form $A\cos 3t + B\sin 3t$. After differentiation we obtain

$$-9A\cos 3t - 9B\sin 3t + 2(-3A\sin 3t + 3B\cos 3t) + 2(A\cos 3t + B\sin 3t) = 4\cos 3t,$$

so $-7A + 6B = 4$ and $-6A - 7B = 0$. Solving this we get the previous result: $A = -\frac{28}{85}$, $B = \frac{24}{85}$.

(c)(3p) Find the general solution of the inhomogeneous equation.

It is $\boxed{y(t) = -\frac{28}{85}\cos 3t + \frac{24}{85}\sin 3t + c_1e^{-t}\cos t + c_2e^{-t}\sin t.}$

(d)(3p) Find the amplitude of the steady-state solution.

It is

$$\boxed{A = \sqrt{\left(\frac{28}{85}\right)^2 + \left(\frac{24}{85}\right)^2} = \frac{\sqrt{1360}}{85}.}$$

6) Consider the system of ordinary differential equations

$$\begin{aligned} x' &= \quad x - xy \\ y' &= \ y - x^2 + 3. \end{aligned}$$

(a)(3p) Find the nullclines and sketch them, clearly indicating which is the x-nullcline and which is the y-nullcline.

x-nullclines: $\boxed{x = 0}$ (the y-axis) and $\boxed{y = 1}$ (a horizontal line).

y-nullclines: $\boxed{y = x^2 - 3}$ (a parabola).

(b)(3p) Find all of the equilibrium points.

We have three eq. points.

$$\boxed{\begin{array}{rcl} x & = & 0 \\ y & = & -3 \end{array}} \quad \boxed{\begin{array}{rcl} x & = & 2 \\ y & = & 1 \end{array}} \quad \boxed{\begin{array}{rcl} x & = & -2 \\ y & = & 1 \end{array}}$$

(c)(3p) Compute the Jacobian matrix.

$$\boxed{J = \begin{bmatrix} 1 - y & -x \\ -2x & 1 \end{bmatrix}.}$$

(d)(3p) Classify the equilibrium points.

At $x = 0, y = -3$ the Jacobian is $J = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$ and we conclude it is a $\boxed{\text{nodal source.}}$

At $x = 2, y = 0$ the Jacobian is $J = \begin{bmatrix} 1 & -2 \\ -4 & 1 \end{bmatrix}$ and we conclude it is a $\boxed{\text{saddle.}}$

At $x = -2, y = 0$ the Jacobian is $J = \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$ and we conclude it is a $\boxed{\text{saddle.}}$

7) Consider the system of ordinary differential equations

$$\begin{aligned} x' &= -x \\ y' &= 2y + x^2. \end{aligned}$$

(a)(3p) Show that the y-axis is invariant.

If we start a solution on the y-axis, then the x-coordinate is zero, $x' = -x = 0$, so the x-coordinate will not change, we will stay on the y-axis.

(b)(3p) Show that for any $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{2t} + \frac{c_1^2}{4}(e^{2t} - e^{-2t}) \end{bmatrix}$$

is the solution of the initial value problem with $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

Clearly $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. Also, $x' = -c_1 e^{-t} = -x$ and $y' = 2c_2 e^{2t} + \frac{c_1^2}{2} e^{2t} + \frac{c_1^2}{2} e^{-2t} = 2y + x^2$.

(c)(4p) Show that the set $S = \{(x, y) : y < -\frac{x^2}{2}\}$ is (positively) invariant.

This is the region under the parabola $y = -\frac{x^2}{2}$. On this parabola, $y' = 0$ by the differential equation and $x' > 0$ when $x < 0$ and $x' < 0$ when $x > 0$, so we cannot leave this region. (Also, $x = y = 0$ is an equilibrium point.)

8) Consider the system of ordinary differential equations

$$\begin{aligned} x' &= x(r^3 - 4r^2 + 4) - y \\ y' &= y(r^3 - 4r^2 + 4) + x, \end{aligned}$$

where $r^2 = x^2 + y^2$ (or $r = \sqrt{x^2 + y^2}$).

(a)(5p) Show that the origin is a spiral source.

Change to polar coordinates. The equations become

$$\begin{aligned} r' = &\ r(r^3 - 4r^2 + 4) \\ \theta' = &\ 1, \end{aligned}$$

This shows that the origin is a spiral source.

Other possibility: linearize the system, the Jacobian is $\begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}$ which shows it is a spiral source again.

(b)(6p) Show that there is a closed solution curve in the annular region

$$A = \{(x, y) : 1 < r < 2\}.$$

From the previous part we can see that this region does not contain an equilibrium point ($\theta' = 1$). Also, the region is positively invariant because $r' = -8 < 0$ when $r = 2$ and $r' = 1 > 0$ when $r = 1$. The assumptions of the Poincaré-Bendixson theorem stand, so the region contains a closed solution curve. (Actually, if we want to be very precise, the assumptions of P-B do not stand, this region is not closed. But we can choose $1 < r_1 < r_2 < 2$ in a way that $r' > 0$ when $r = r_1$ and $r' < 0$ when $r = r_2$ and then apply P-B for $A_1 = \{(x, y) : r_1 \le r \le r_2\} \subset A$.)

2nd Solution. For (b): from the polar coordinate form it is clear that there is a value r^* between 1 and 2 such that $r' = 0$ if $r(t) = r^*$. This, together with the equation for θ' proves the statement.

9) Consider the matrix

$$A = \begin{bmatrix} 2 & 7 & 2 \\ 1 & 2 & 0 \\ 0 & 3 & 2 \end{bmatrix}.$$

(a)(4p) Find the nullspace of A.

One possible echelon form of the matrix:

$$A \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

We have one free variable, and we obtain that the nullspace is

$$\boxed{\text{null }(A) = \text{span}\{\begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}\}.}$$

(b)(4p) What is $\det(A)$?

According to (a), A is singular, so its $\boxed{\text{determinant is } 0.}$

(c)(4p) Are the column vectors in A linearly independent?

According to (a), A is singular, so they are $\boxed{\text{not linearly independent.}}$