Math 211
Final
December 2001

Make sure to show your work and justify your arguments.

Calculator policy: You may use calculators to evaluate standard functions on floating point numbers (like $\sqrt{3.12}$, $\ln(35/7)$, or $\sin(\pi/17)$). You may not use symbolic operations, numerical integration, or any graphing functions.

1) Consider the ordinary differential equation

$$yy' = 2t\sqrt{y^2+4}.$$

(a)(4p) Find the general solution.

(b)(4p) Find the solution of the initial value problem when $y(2) = -\sqrt{5}$.

(c)(4p) Find the interval of existence of this solution.

2)(10p) Find the general solution of the ordinary differential equation.

$$t^2y' + 3ty = \frac{\sin t}{t} \qquad (t > 0).$$

3)(10p) Three populations exist together and interact in isolated circumstances. Denote the populations by $x_1(t)$, $x_2(t)$ and $x_3(t)$. They interact as follows:

- The first population preys upon the other two and would decline dramatically if $x_2 = x_3 = 0$.
- The second population preys upon the third and would be able to survive if the third were not present, but its growth would be limited by the resources.
- The third population would flourish in the absence of the other two, but its growth would be limited by the resources.

Model the interactions between the three populations with a system of ordinary differential equations. You are **not** required to solve the equations.

4) Consider the system of ordinary differential equations

$$y' = Ay, \text{ where } A = \begin{bmatrix} -1 & 1 & 1 \\ -4 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a)(3p) Find a fundamental set of solutions.

(b)(3p) Find the general solution of the system.

(c)(3p) Find the solution of the initial value problem when $y(0) = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$.

(d)(2p) What can we say about the stability of the equilibrium point at the origin?

5) Consider the periodically forced harmonic ordinary differential equation

$$y'' + 2y' + 2y = 4\cos 3t.$$

(a)(3p) Find a fundamental set of solutions of the associated homogeneous equation.

(b)(3p) What is the steady-state solution?

(c)(3p) Find the general solution of the inhomogeneous equation.

(d)(3p) Find the amplitude of the steady-state solution.

6) Consider the system of ordinary differential equations

$$\begin{aligned} x' &= \quad x - xy \\ y' &= \quad y - x^2 + 3. \end{aligned}$$

(a)(3p) Find the nullclines and sketch them, clearly indicating which is the x-nullcline and which is the y-nullcline.

(b)(3p) Find all of the equilibrium points.

(c)(3p) Compute the Jacobian matrix.

(d)(3p) Classify the equilibrium points.

7) Consider the system of ordinary differential equations

$$\begin{aligned} x' &= \quad -x \\ y' &= \quad 2y + x^2. \end{aligned}$$

(a)(3p) Show that the y-axis is invariant.

(b)(3p) Show that for any $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{2t} + \frac{c_1^2}{4}(e^{2t} - e^{-2t}) \end{bmatrix}$$

is the solution of the initial value problem with $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

(c)(4p) Show that the set $S = \{(x, y) : y < -\frac{x^2}{2}\}$ is (positively) invariant.

8) Consider the system of ordinary differential equations

$$\begin{aligned} x' &= x(r^3 - 4r^2 + 4) - y \\ y' &= y(r^3 - 4r^2 + 4) + x, \end{aligned}$$

where $r^2 = x^2 + y^2$ (or $r = \sqrt{x^2 + y^2}$).

(a)(5p) Show that the origin is a spiral source.

(b)(6p) Show that there is a closed solution curve in the annular region

$$A = \{(x, y) : 1 < r < 2\}.$$

9) Consider the matrix

$$A = \begin{bmatrix} 2 & 7 & 2 \\ 1 & 2 & 0 \\ 0 & 3 & 2 \end{bmatrix}.$$

(a)(4p) Find the nullspace of A.

(b)(4p) What is $\det(A)$?

(c)(4p) Are the column vectors in A linearly independent?