Math 211
Second Midterm
April 1, 2003

Make sure to show your work and justify your arguments.

Calculator policy: You may NOT use a calculator on this exam. The reasons are twofold: first, the calculations are not hard and you will be able to do them in your head. Second, some calculators are equipped with programs which evaluate determinants and do matrix operations.

1) (15p) Find the solution of the linear system of equations

$$\begin{aligned} x + 2y - 3z + 5v + 3w &= 10 \\ 2x + 4y - 6z + 11v + 8w &= 23 \\ -x - 2y + 3z - 3v + w &= -4. \end{aligned}$$

Help: using MATLAB, the reduced row echelon form of the augmented matrix of this system is

$$\begin{bmatrix} 1 & 2 & -3 & 0 & -7 & -5 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution The pivot variables are x and v, the others are free. Let $y = r$, $z = s$ and $w = t$. Solving for x and v we obtain that $x = -5 - 2r + 3s + 7t$ and $v = 3 - 2t$ so we can write the solution in the form

$$\begin{bmatrix} x \\ y \\ z \\ v \\ w \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} + r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 7 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

2) (20p) Consider the system of linear equations

$$\begin{aligned} kx + y + z &= 1 \\ x + ky + z &= 1 \\ x + y + kz &= 1. \end{aligned}$$

(k is a real constant.) For which values of k does the system have (a) a unique solution, (b) no solution, (c) infinitely many solutions?

Solution The determinant of the coefficient matrix is $k^3 - 3k + 2 = (k - 1)(k^2 + k - 2) = (k - 1)^2(k + 2)$. If the determinant is not zero, the inverse exists and the solution is unique. This is the case when $k \neq 1$ and $k \neq -2$. If $k = 1$, then the three equations are the same and we have infinitely many solutions. If $k = -2$, then adding the three equations we obtain $0 = 3$ which is impossible, so we do not have a solution.

3) (20p) Find those x values for which the matrix

$$\begin{bmatrix} 2 & -1 & 3 \\ 5 & -1 & x \\ x & 1 & -1 \end{bmatrix}$$

is nonsingular.

Solution The determinant is $-x^2+x+12 = -(x-4)(x+3)$. This is nonzero when $x \neq 4$ and $x \neq -3$. So for these (infinitely many) values the matrix is nonsingular.

4) Consider the system of ODE's

$$\begin{aligned} x' &= x - x^4 + y^2 \\ y' &= -2x^2 + 4y. \end{aligned}$$

(a)(4p) Verify that $x = e^t$, $y = e^{2t}$ is a solution.
(b)(4p) Verify that $x = -e^t$, $y = e^{2t}$ is a solution.
(c)(7p) Consider the solution to the system with initial conditions $x(0) = 0$, $y(0) = 1$. Is there a time T such that $x(T) = 2$ and $y(T) = 1$? Explain.

Solution (a) and (b): just plug them in.

(c): The curves in (a) and (b) are two halves of the curve $y = x^2$. $(0,0)$ is also a solution. So the curve $y = x^2$ is made up of three solutions of the ODE. The solution which starts at the point $(0,1)$ cannot intersect this curve, so we cannot reach the point $(2,1)$.

5) (15p) Consider the three vectors

$$\begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} \quad \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}.$$

Are these vectors linearly independent? If not, express the zero vector as a nontrivial linear combination of these vectors.

Solution Yes, they are. Observe that if you form a 3×3 matrix from these vectors, you obtain the matrix from problem 3 with $x = 2$. But for that value the matrix is nonsingular. This implies these vectors are linearly independent.

6) (15p) Check that $\det(A)\det(B) = \det(AB)$ for the matrices

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & 2 \\ 0 & -1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 4 \\ 1 & 1 & 1 \end{bmatrix}.$$

Solution $\det(A) = 5$, $\det(B) = -5$ and $\det(AB) = -25$.

$$AB = \begin{bmatrix} 2 & 7 & -3 \\ 8 & 8 & 3 \\ -1 & 2 & -2 \end{bmatrix}.$$