Math 211, Exam 2
November 14th, 2002

1. (a) **(15 points)** Find a parametric representation for the solution set of the system

$$x_1 + 2x_2 - 2x_3 + x_4 = 2$$
$$x_2 - 3x_3 - x_4 = 3$$
$$x_3 - x_4 = 0$$

(b) How would you (geometrically) describe the solution set? (Examples of geometrical description include a circle in $\mathbb{R}^2$, a line in $\mathbb{R}^3$, a hyperbola in $\mathbb{R}^2$, a plane in $\mathbb{R}^4$ and an empty set.)

Answer:

(a) The augmented matrix for the system is

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 2 \\ 0 & 1 & -3 & -1 & 3 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}$$

There is a free variable: Let $x_4 = t$. Then

$$x_3 = x_4 = t$$
$$x_2 = 3 + x_4 + 3x_3 = 3 + 4t$$
$$x_1 = 2 - 2x_2 + x_3 - x_4 = -4 - 8t$$

Therefore,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -4 - 8t \\ 3 + 4t \\ t \\ t \end{pmatrix} = t \begin{pmatrix} -8 \\ 4 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -4 \\ 3 \\ 0 \\ 0 \end{pmatrix}.$$

(b) This is a parametric representation of a line in $\mathbb{R}$.

2. **(15 points)** Which of the following matrices
   (a) is(are) singular?
   (b) have non-trivial nullspace?
   (c) have determinant $= 0$?

   Justify your answers.

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 2 & 5 & -3 \\ -3 & 2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -6 \\ -12 & 24 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 2 & 2 & 0 & 1 & 2 \\ 1 & 2 & 0 & 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Answer: All parts : B, C

(a) B has two rows that are constant multiples of each other, and C has two equal rows.

(b) The matrix B is equivalent to $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ in the matrix equation : $B\mathbf{x} = \mathbf{z}$, and has nontrivial nullspace. The matric C is equivalent to a 7×7 matrix with a row of zeros, and thus has nontrivial nullspace.

(c)

$$\det(B) = \frac{1}{3} \det \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = 0, \quad \det(C) = \det \begin{pmatrix} 0 & 1 & \cdots & 1 & 2 \\ & \vdots & & & \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} = 0.$$

3. **(10 points)** Write the third order differential equation

$$y''' + y'y'' = \sin \omega t$$

with initial values

$$y(0) = 1,\ y'(0) = -1,\ y''(0) = 0$$

as a system of first order differential equations (DO NOT solve the system).

Answer:

Let $u_1 = y,\ u_2 = y',\ u_3 = y''$. Then we have the system

$$\begin{aligned} u_1' &= u_2, \\ u_2' &= u_3, \\ u_3' &= \sin \omega t - u_2 u_3, \end{aligned}$$

with

$$\mathbf{u}(0) = (1, -1, 0)^T.$$

4. **(20 points)** Consider the system

$$x' = y - x(x^2 + y^2 - 1)$$
$$y' = -x - y(x^2 + y^2 - 1).$$

(a) Show that $x(t) = \sin t,\ y(t) = \cos t$ is a solution.
(b) Sketch the solution in part(a) in the phase plane.
(c) Consider the solution to the system with the initial conditions $x(0) = 0.5$ and $y(0) = 0$. Show that $x^2(t) + y^2(t) < 1$ for all t.

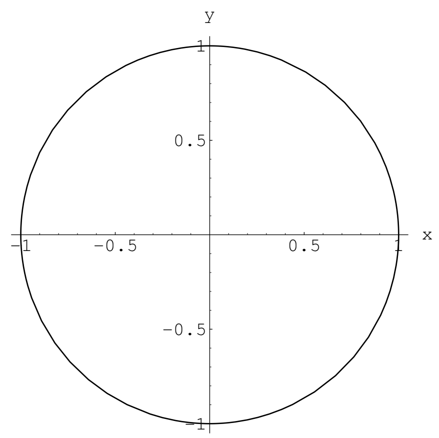
Answer:

(a) $x(t) = \sin t \Rightarrow x'(t) = \cos t,\quad y(t) = \cos t \Rightarrow y'(t) = -\sin t$. On the other hand,

$$y - x(x^2 + y^2 - 1) = \cos t - \sin t(\sin^2 t + \cos^2 = 1) = \cos t,$$
$$-x - y(x^2 + y^2 - 1) = -\sin t - \cos t(\sin^2 t + \cos^2 = 1) = \cos t.$$

Thus, $(x(t), y(t))^T$ is a solution.

(b) The parametric curve for $(\sin t, \cos t)$ is a circle of radius 1, centered at 0.



(c) The RHS of the differential equation is continuous and is differentiable with respect to both x and y. Thus the uniqueness theorem applies: solutions to the differential equation cannot intersect. The solution with $(x(0), (y(0)) = (0.5, 0)$ starts inside the circle, and it would have to cross the circle if $x(t)^2 + y(t)^2 \geq 1$ for any t. Thus, $x(t)^2 + y(t)^2 < 1$ for all t.

5. **(20 points)**

(a) Find the general solution to $\mathbf{y}' = A\mathbf{y}$ for

$$A = \begin{pmatrix} 2 & -6 \\ 0 & -1 \end{pmatrix}.$$

(b) Find all equilibrium points and plot them in the phase plane. Plot the half-line solutions (exponential solutions) and sketch enough solution curves to show what happens in every part of the phase plane. Use arrows to indicate direction of motion for all solutions.

(c) Classify the equilibrium points as one of the 6 types discussed in class.

Answer:

(a) The characteristic polynomial of$A$ is $(\lambda - 2)(\lambda + 1)$, so the eigenvalues are $\lambda_1 = -1,\ \lambda_2 = 2.$

<u>For $\lambda_1 = -1$</u>

$$A - \lambda_1 I = \begin{pmatrix} 3 & -6 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \Rightarrow$$

nullspace of $A + I$ is spanned by the eigenvector $\mathbf{v}_1 = (2, 1)^T$.

<u>For $\lambda_2 = 2$</u>

$$A - \lambda_2 I = \begin{pmatrix} 0 & -6 \\ 0 & -3 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow$$

nullspace of $A - 2I$ is spanned by the eigenvector $\mathbf{v}_1 = (1, 0)^T$.

The general solution:

$$\mathbf{x}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

(b) Equilibrium points:

$$A\mathbf{x} = \mathbf{z} \Leftrightarrow \mathbf{x} = \mathbf{z} \Rightarrow \mathbf{z} \text{ is the only equilibrium point.}$$

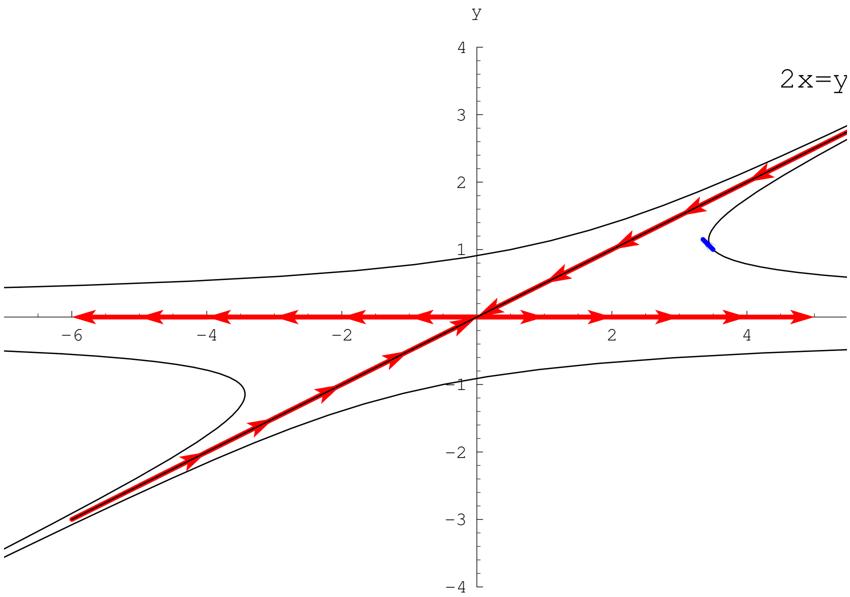
Half-line solutions are

$$y = 0, \quad 2x = y.$$

(c) Since we have real eigenvalues with

$$\lambda_1 < 0 < \lambda_2,$$

we have a saddle.



6. **(20 points)** Consider the 3×3 matrix

$$A = \begin{pmatrix} -4 & 8 & 8 \\ -4 & 4 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

(a) A has a complex eigenvalue $\lambda = 4i$, with its associated eigenvector $\mathbf{v} = (1-i, 1, 0)^T$. Find **all** eigenvalues and associated eigenvectors.

(b) Find a fundamental set of real solutions to the system $\mathbf{x}' = A\mathbf{x}$.

(c) Solve the initial value problem

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = (4, -1, 2)^T.$$

Answer:

(a) $\det(A - \lambda I) = (2-\lambda)[(\lambda-4)(\lambda+4)+32] \Rightarrow \lambda_1 = 2,\ \lambda_2 = 4i,\ \lambda_3 = -4i.$

For $\lambda_1 = 2$,

$$A - 2I = \begin{pmatrix} -6 & 8 & 8 \\ -4 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & -1 \\ -3 & 4 & 4 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} -2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \text{null}(A - 2I) = t\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 4i,\ \lambda_3 = -4i$

$$\mathbf{v}_2 = \begin{pmatrix} 1-i \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1+i \\ 1 \\ 0 \end{pmatrix}$$

(b) We have $\mathbf{x}_1(t) = e^2t(0, -1, 1)^T$.

For complex eigenvalue $\lambda_2 = 4i$, we have the complex solution

$$z(t) = e^{4it}\begin{pmatrix} 1-i \\ 1 \\ 0 \end{pmatrix} = (\cos 4t + i\sin 4t)\left[\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}\right]$$

$$= \left[\cos 4t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \sin 4t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right] + i\left[\cos 4t\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \sin 4t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right].$$

Thus, let

$$\mathbf{x}_2(t) = \begin{pmatrix} \cos 4t + \sin 4t \\ \cos 4t \\ 0 \end{pmatrix}, \quad \mathbf{x}_3(t) = \begin{pmatrix} -\cos 4t + \sin 4t \\ \sin 4t \\ 0 \end{pmatrix}.$$

The general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + c_3\mathbf{x}_3(t) \\ &= c_1 e^{2t} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} \cos 4t + \sin 4t \\ \cos 4t \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -\cos 4t + \sin 4t \\ \sin 4t \\ 0 \end{pmatrix}. \end{aligned}$$

(c) Initial value: $\mathbf{x}(0) = (4, -1, 2)^T$. Thus we look for $c_1,\ c_2,\ c_3$ that satisfy

$$c_1 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}.$$

Solving the linear system with the augmented matrix

$$\begin{pmatrix} 0 & 1 & -1 & 4 \\ -1 & 1 & 0 & -1 \\ 1 & 0 & - & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 3 \end{pmatrix} \Rightarrow c_1 = 2,\ c_2 = 1,\ c_3 = -3.$$

Therefore,

$$\begin{aligned} \mathbf{x}(t) &= c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + c_3\mathbf{x}_3(t) \\ &= 2e^{2t} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} \cos 4t + \sin 4t \\ \cos 4t \\ 0 \end{pmatrix} - 3 \begin{pmatrix} -\cos 4t + \sin 4t \\ \sin 4t \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 2e^{2t} + 4\cos 4t - 2\sin 4t \\ -2e^{2t} + \cos 4t - 3\sin 4t \\ 2e^{2t} \end{pmatrix} \end{aligned}$$