

Math 211 Exam 2 with Solution

November 13, 2003

1. (20 points) Let

$$A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

- (a) What is the dimension of $\text{null}(A)$?
- (b) Find a basis for $\text{null}(A)$.
- (c) Suppose \mathbf{x}_p is a particular solution to $A\mathbf{x} = \mathbf{b}$ for some $\mathbf{b} \in \mathbf{R}^4$, find the general solution to $A\mathbf{x} = \mathbf{b}$.

Solution.

- (a) Note that A is already in the row echelon form which has one free column. So $\dim(\text{null}(A)) = 1$.
- (b) Let $x_4 = 1$, back-solve the system to get

$$\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

which form a basis for $\text{null}(A)$.

- (c) The general solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x}_p + t\mathbf{v}$, $t \in \mathbf{R}$.

2. (20 points)

- (a) Find those x values for which the following vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ x \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} x \\ 1 \\ 1 \end{bmatrix}$$

are linearly dependent.

- (b) For each x found in part (a), express the zero vector as a nontrivial linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Solution.

- (a) It suffices to find those x values such that the matrix

$$V = \begin{bmatrix} 1 & 1 & x \\ 1 & x & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

is singular or equivalently, $\det(A) = 0$.

Method 1: Since

$$\begin{bmatrix} 1 & 1 & x \\ 1 & x & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[\substack{R2 \rightarrow R1 \\ R3 \rightarrow R1}]{R2 \rightarrow R1} \begin{bmatrix} 1 & 1 & x \\ 0 & x-1 & 1-x \\ 0 & 0 & 1-x \end{bmatrix}$$

V is singular if and only if $x - 1 = 0$, i.e. $x = 1$.

Method 2:

$$\det \begin{bmatrix} 1 & 1 & x \\ 1 & x & 1 \\ 1 & 1 & 1 \end{bmatrix} = -x^2 + 2x - 1 = -(x-1)^2 = 0 \Leftrightarrow x = 1.$$

So, $x = 1$ is the only value for which the vectors are linearly dependent.

(b) When $x = 1$,

$$\begin{bmatrix} 1 & 1 & x \\ 0 & x-1 & 1-x \\ 0 & 0 & 1-x \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

There are two free columns in the equivalent row echelon form of V . Choose $c_2 = s$, $c_3 = t$, to find $c_1 = -s - t$. So, for any $s, t \in \mathbf{R} \setminus \{0\}$, $-(s+t)\mathbf{v}_1 + s\mathbf{v}_2 + t\mathbf{v}_3 = \mathbf{0}$.

3. (20 points) Consider the system

$$\begin{aligned} x' &= 1 - (y - \sin x) \cos x \\ y' &= \cos x - y + \sin x. \end{aligned}$$

- (a) Show that $x(t) = t$, $y(t) = \sin t$ is a solution.
- (b) Plot the solution found in part (a) in the phase plane.
- (c) Consider the solution to the system with the initial conditions $x(0) = \pi/2$ and $y(0) = 0$. Show that $y(t) < \sin x(t)$ for all t .

Solution.

(a) Suppose $x(t) = t$, $y(t) = \sin t$. For the first equation

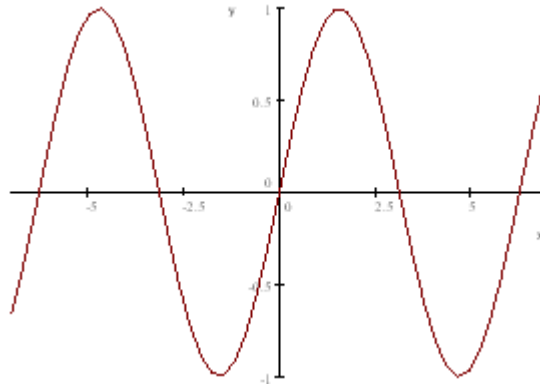
$$\begin{aligned} LHS &= x'(t) = 1, \\ RHS &= 1 - (\sin t - \sin t) \cos t = 1, \\ RHS &= LHS. \end{aligned}$$

For the second equation

$$\begin{aligned} LHS &= y'(t) = \cos t, \\ RHS &= \cos t - \sin t + \sin t = \cos t, \\ RHS &= LHS. \end{aligned}$$

So $x(t) = t$, $y(t) = \sin t$ is a solution.

(b) In the phase plane, we have $y = \sin x$,



(c) Since the system is autonomous and the derivative functions are continuous with continuous derivatives, the solution curves in the phase plane can't meet at a point unless the curves coincide. Since $y(0) < \sin x(0)$, we have $y(t) < \sin x(t)$ for all t .

4. (20 points)

(a) Find the general solution to

$$\mathbf{y}' = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \mathbf{y}.$$

(b) Find all equilibrium points and plot them in the phase plane and sketch the solution curves. Use arrows to indicate direction of motion for all solutions.

(c) Classify the equilibrium points as one of the 6 types discussed in class.

(d) Classify the equilibrium points as unstable, stable but not asymptotically stable, or asymptotically stable.

Solution.

(a) The characteristic polynomial

$$p(\lambda) = \det \begin{bmatrix} 1 - \lambda & -2 \\ 1 & -1 - \lambda \end{bmatrix} = 1 + \lambda^2.$$

Solve $p(\lambda) = 0$ to get the eigenvalues $\lambda = \pm i$.

$$\begin{bmatrix} 1 - i & -2 \\ 1 & -1 - i \end{bmatrix} \xrightarrow{\text{row operation}} \begin{bmatrix} 1 - i & -2 \\ 0 & 0 \end{bmatrix}.$$

An eigenvector for $\lambda = i$ is

$$\begin{bmatrix} 2 \\ 1 - i \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} i.$$

The general solution is

$$\begin{aligned} \mathbf{y}(t) &= c_1 e^{0t} (\cos(t) \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \sin(t) \begin{bmatrix} 0 \\ -1 \end{bmatrix}) + c_2 e^{0t} (\sin(t) \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \cos(t) \begin{bmatrix} 0 \\ -1 \end{bmatrix}) \\ &= \begin{bmatrix} 2c_1 \cos t + 2c_2 \sin t \\ (c_1 - c_2) \cos t + (c_1 + c_2) \sin t \end{bmatrix}. \end{aligned}$$

- (b) Since $\det \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \neq 0$, $\mathbf{0}$ is the only equilibrium point. Since the eigenvalues are purely imaginary, $\mathbf{0}$ is a center. We can find the spiral direction by finding

$$\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

So the solutions go counterclockwise.

- (c) Center.
(d) Stable but not asymptotically stable.

5. (20 points) Consider the 3×3 matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

- (a) Find a fundamental set of solutions to the system $\mathbf{x}' = A\mathbf{x}$.
(b) Solve the initial problem $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = (1, 1, 1)^T$.

Solution.

- (a) Find and factorize the characteristic polynomial to get

$$p(\lambda) = (\lambda - 1)(\lambda - 2)^2.$$

A has two eigenvalues $\lambda_1 = 1$ with algebraic multiplicity 1, $\lambda_2 = 2$ with algebraic multiplicity 2.

$$A - I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

the vector $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector associated with 1. The corresponding solution is

$$\mathbf{x}_1(t) = e^{tA}\mathbf{v}_1 = e^t\mathbf{v}_1 = \begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ has nullspace of dimension 1,}$$

$$(A - 2I)^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ has nullspace of dimension 2.}$$

$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis of $\text{null}(A - I)^2$, with corresponding solutions

$$\mathbf{x}_2(t) = e^{tA}\mathbf{v}_2 = e^{2t}(\mathbf{v}_2 + t(A - 2I)\mathbf{v}_2) = \begin{bmatrix} 0 \\ e^{2t} \\ 0 \end{bmatrix},$$

$$\mathbf{x}_3(t) = e^{tA}\mathbf{v}_3 = e^{2t}(\mathbf{v}_3 + t(A - 2I)\mathbf{v}_3) = \begin{bmatrix} 0 \\ te^{2t} \\ e^{2t} \end{bmatrix}.$$

A fundamental set of solutions is

$$\left\{ \begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ e^{2t} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ te^{2t} \\ e^{2t} \end{bmatrix} \right\}.$$

(b) Since the general solution can be written as $\mathbf{x}(t) = c_1 \begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{2t} \\ 0 \end{bmatrix} +$

$c_3 \begin{bmatrix} 0 \\ te^{2t} \\ e^{2t} \end{bmatrix}$, solve $\mathbf{x}(0) = (1, 1, 1)^T$ to get $c_1 = c_2 = c_3 = 1$. The solution for the given I.V.P. is

$$\mathbf{x}(t) = \begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ e^{2t} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ te^{2t} \\ e^{2t} \end{bmatrix} = \begin{bmatrix} e^t \\ (1+t)e^{2t} \\ e^{2t} \end{bmatrix}.$$