Math 211

Exam # 2

April 3, 2001

1. Consider the system of differential equations

$$\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= -x_1 + t^2. \end{aligned}$$

a) (5 points) Is the pair of functions $x_1(t) = t^2$, $x_2(t) = t - 3$ a solution?

Answer: We compute that $x_1' = 2t$ while $x_1 + x_2 = t^2 + t - 3$. Since these are not equal this pair is not a solution.

b) (5 points) Is the pair of functions $x_1(t) = t^2 + 2t$, $x_2(t) = -t^2 + 2$ a solution?

Answer: Now $x_1' = 2t + 2$, and $x_1 + x_2 = t^2 + 2t - t^2 + 2 = 2t + 2$. Next $x_2' = -2t$, and $-x_1 + t^2 = t^2 + 2t - t^2 = -2t$. Since we have equality in both cases, this pair is a solution.

2. (10 points) Is the matrix

$$A = \begin{pmatrix} 3 & -2 \\ -1 & 0 \end{pmatrix}$$

singular or nonsingular? Is invertible? Does it have a nontrivial nullspace?

Answer: We compute that $\det(A) = 2$. Since this is not equal to 0, A is nonsingular. It is therefore invertible, and its nullspace is trivial.

3. Consider the system $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}.$$

a) (5 points) Show that

$$\mathbf{y}_1(t) = e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{y}_2(t) = e^{-t}\begin{pmatrix} 1-t \\ -t \end{pmatrix}$$

are both solutions.

Answer: We have

$$\mathbf{y}_1' = -e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad A\mathbf{y} = e^{-t}\begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = e^{-t}\begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

Since these are equal, $\mathbf{y}_1$ is a solution. Similarly

$$\mathbf{y}_2' = -e^{-t}\begin{pmatrix} 1-t \\ -t \end{pmatrix} + e^{-t}\begin{pmatrix} -1 \\ -1 \end{pmatrix} = e^{-t}\begin{pmatrix} t-2 \\ t-1 \end{pmatrix}, \quad \text{and}$$

$$A\mathbf{y}_2 = e^{-t}\begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1-t \\ -t \end{pmatrix} = e^{-t}\begin{pmatrix} t-2 \\ t-1 \end{pmatrix}.$$

Since these are equal, $\mathbf{y}_2$ is also a solution.

b) (10 points) Assuming they are solutions, show that $\mathbf{y}_1$ and $\mathbf{y}_2$ form a fundamental set of solutions.

Answer: We have

$$\mathbf{y}_1(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{y}_2(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

since these vectors are not multiples of each other, they are linearly independent. Consequently the solutions $\mathbf{y}_1$ and $\mathbf{y}_2$ are also linearly independent and therefore are a fundamental set of solutions.

c) (10 points) Assuming the answer to part b), find the solution to the initial value problem $\mathbf{y}' = A\mathbf{y}$, with $\mathbf{y}(0) = (1, 5)^T$.

Answer: Since $\mathbf{y}_1$ and $\mathbf{y}_2$ are a fundamental set of solutions, we know that $\mathbf{y} = C_1\mathbf{y}_1 + C_2\mathbf{y}_2$ for some constants C_1 and C_2. The initial conditions implies that $\mathbf{y}(0) = C_1\mathbf{y}_1(0) + C_2\mathbf{y}_2(0)$, or

$$\begin{pmatrix} 1 \\ 5 \end{pmatrix} = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

solving, we find that $C_1 = 5$ and $C_2 = -4$. Hence our solution is

$$\begin{aligned} \mathbf{y}(t) &= C_1\mathbf{y}_1(t) + C_2\mathbf{y}_2(t) \\ &= 5e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 4e^{-t}\begin{pmatrix} 1-t \\ -t \end{pmatrix} \\ &= e^{-t}\begin{pmatrix} 1+4t \\ 5+4t \end{pmatrix}. \end{aligned}$$

4. Consider the system

$$\begin{aligned} x' &= -y + x(4 - x^2 - y^2) \\ y' &= x + y(4 - x^2 - y^2) \end{aligned}$$

a) (5 points) Show that $x(t) = 2\cos t$, $y(t) = 2\sin t$ is a solution.

Answer: In the first equation we have $x' = -2\sin t$ on the left and

$$-y + x(4 - x^2 - y^2) = -2\sin t + 2\cos t(4 - 4\cos^2 t - 4\sin^2 t) = -2\sin t$$

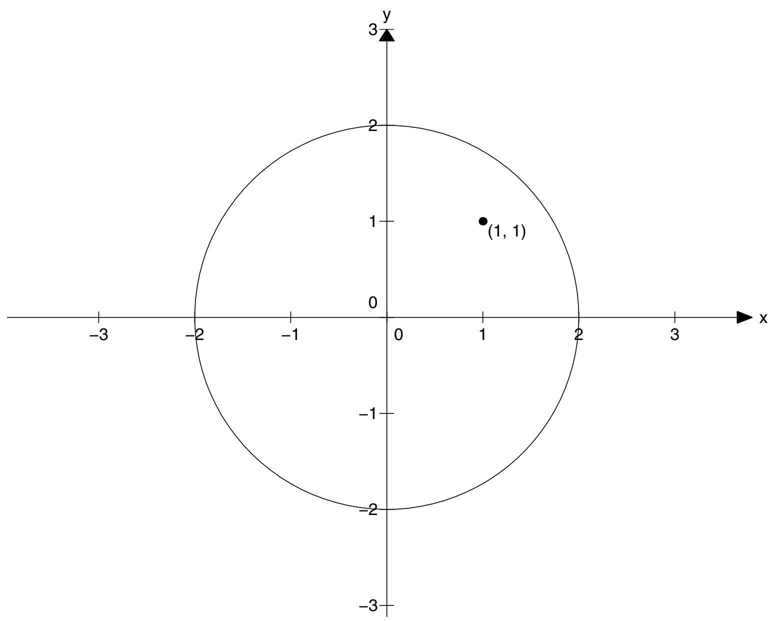
on the right. Thus the first equation is satisfied. For the second equation we have $y' = 2\cos t$ on the left and

$$x + y(4 - x^2 - y^2) = 2\cos t + 2\sin t(4 - 4\cos^2 t - 4\sin^2 t) = 2\cos t$$

on the right. Again the equation is satisfied.

b) (5 points) Sketch the solution found in part a) in the phase plane.

Answer: The solution curve is the circle of radius 2 plotted in the following figure.



The solution curve in part a) and the initial point in part c).

c) (5 points) Suppose we have another solution $x(t)$, $y(t)$ which satisfies the initial conditions $x(0) = 1$ and $y(0) = 1$. Show that $x^2(t) + y^2(t) < 4$ for all t.

Answer: The initial point is shown in the figure above. The solution curve starting there cannot cross the solution curve in part a). Thus it must remain inside the circle of radius 2, which means that $x^2(t) + y^2(t) < 4$ for all t.

5. (10 points) Describe the type (i.e., saddle point, nodal source, etc.) of the equilibrium point at $\mathbf{0}$ for the system $\mathbf{x}' = A\mathbf{x}$ in each of the following cases

a) $A = \begin{pmatrix} 5 & 6 \\ -4 & -5 \end{pmatrix}$

Answer: We have determinant $D = -1$. Since the determinant is negative, the equilibrium point is a saddle.

b) $A = \begin{pmatrix} -1 & -4 \\ 2 & 5 \end{pmatrix}$

Answer: We have trace $T = 4$, and determinant $D = 3$. The discriminant $T^2 - 4D = 4$ is positive. Hence the eigenvalues are real. The equilibrium point is a nodal source.

c) $A = \begin{pmatrix} -5 & -3 \\ 6 & 1 \end{pmatrix}$

Answer: We have trace $T = -4$ and determinant $D = 13$. The discriminant $T^2 - 4D = -36$ is negative. Hence the eigenvalues are complex with negative real part. The equilibrium point is a spiral sink.

d) $A = \begin{pmatrix} -5 & -5 \\ 10 & 5 \end{pmatrix}$

Answer: The trace $T = 0$ and the determinant $D = 25$ is positive. Hence the equilibrium point is a center.

6. Consider the system $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{pmatrix} -2 & 0 & 0 \\ 2 & -3 & -2 \\ -2 & 4 & 3 \end{pmatrix}.$$

a) (10 points) Describe the main steps in procedure that you follow to find a fundamental set of solutions for a system like this.

Answer: The main steps are:

- Compute the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$.
- Find the eigenvalues, which are the roots of the characteristic polynomial.
- For each eigenvalue λ find the eigenspace, which is equal to the nullspace of $A - \lambda I$.
- Make sure that we have n (3 in this case) linearly independent solutions.

b) (20 points) Find a fundamental set of solutions. Be sure to point out why they form a fundamental set of solutions.

Answer: We compute the characteristic polynomial by expanding the determinant along the first row. We get

$$
\begin{aligned}
p(\lambda) &= \det(A - \lambda I) \\
&= \det \begin{pmatrix} -2-\lambda & 0 & 0 \\ 2 & -3-\lambda & -2 \\ -2 & 4 & 3-\lambda \end{pmatrix} \\
&= (-2-\lambda) \det \begin{pmatrix} -3-\lambda & -2 \\ 4 & 3-\lambda \end{pmatrix} \\
&= -(\lambda+2)[(-3-\lambda)(3-\lambda)+8] \\
&= -(\lambda+2)[\lambda^2 - 1] \\
&= -(\lambda+2)(\lambda+1)(\lambda-1).
\end{aligned}
$$

Thus the eigenvalues are -1, -2, and 1.

For the eigenvalue $\lambda_1 = -2$ we have

$$A - \lambda_1 I = A + 2I = \begin{pmatrix} 0 & 0 & 0 \\ 2 & -1 & -2 \\ -2 & 4 & 5 \end{pmatrix}.$$

Using row operations we can reduce this to the row echelon form

$$\begin{pmatrix} 2 & -1 & -2 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

By backsolving we see that $\mathbf{v}_1 = (1, -2, 2)^T$ is in the nullspace and therefore is an eigenvector associated with $\lambda_1 = -2$. Hence we have the exponential solution

$$\mathbf{y}_1(t) = e^{\lambda_1 t}\mathbf{v}_1 = e^{-2t} \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}.$$

For the eigenvalue $\lambda_2 = -1$ we have

$$A - \lambda_2 I = A + I = \begin{pmatrix} -1 & 0 & 0 \\ 2 & -2 & -2 \\ -2 & 4 & 4 \end{pmatrix}.$$

Using row operations we can reduce this to the row echelon form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{pmatrix}.$$

By backsolving we see that $\mathbf{v}_2 = (0, -1, 1)^T$ is in the nullspace and therefore is an eigenvector associated with $\lambda_2 = -1$. Hence we have the exponential solution

$$\mathbf{y}_2(t) = e^{\lambda_2 t}\mathbf{v}_2 = e^{-t} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

For the eigenvalue $\lambda_3 = 1$ we have

$$A - \lambda_3 I = A - I = \begin{pmatrix} -3 & 0 & 0 \\ 2 & -4 & -2 \\ -2 & 4 & 2 \end{pmatrix}.$$

Using row operations we can reduce this to the row echelon form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

By backsolving we see that $\mathbf{v}_3 = (0, 1, -2)^T$ is in the nullspace and therefore is an eigenvector associated with $\lambda_3 = 1$. Hence we have the exponential solution

$$\mathbf{y}_3(t) = e^{\lambda_3 t}\mathbf{v}_3 = e^t \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}.$$

Since the three eigenvalues are distinct, the eigenvectors are linearly independent and so are the exponential solutions. Hence $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ form a fundamental set of solutions.