Math 211
Second Midterm
October 30, 2001
Solutions

1)(14p) Find the nullspace of the matrix

$$A = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & 0 & 2 \end{bmatrix}.$$

Solution

Elementary row operations:

$$A = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & 0 & 2 \end{bmatrix} \xrightarrow{R2-R1/2} \begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1/2 \\ 0 & 4 & 0 & 2 \end{bmatrix} \xrightarrow{R3-2R1} \begin{bmatrix} 0 & \boxed{2} & 0 & 1 \\ 0 & 0 & \boxed{1} & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

If we denote the variables by x, y, z and t, then y and z are pivot variables, x and t are free variables. So to find a basis for the nullspace first we put $x = 1$ and $t = 0$ and solve for y and z, then we put $x = 0$ and $t = 1$ and then solve for y and z. The results:

$$v_1 = \begin{bmatrix} 0 \\ -1/2 \\ 1/2 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is a basis for the nullspace of A.

Alternative way: set the free variables as parameters, solve for the pivot variables and separate the vectors.

2) Consider the vectors $u = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $v = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$.

a)(7p) Write $w = \begin{bmatrix} 8 \\ -1 \\ 5 \end{bmatrix}$ as a linear combination of u and v.

Solution

$$A = \begin{bmatrix} 2 & -1 & 8 \\ -1 & -1 & -1 \\ 3 & 2 & 5 \end{bmatrix} \xrightarrow{R2 \leftrightarrow R1} \begin{bmatrix} -1 & -1 & -1 \\ 2 & -1 & 8 \\ 3 & 2 & 5 \end{bmatrix} \xrightarrow{R2+2R1, R3+R1} \begin{bmatrix} -1 & -1 & -1 \\ 0 & -3 & 6 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\xrightarrow{R3-R2/3} \begin{bmatrix} \boxed{-1} & -1 & -1 \\ 0 & \boxed{-3} & 6 \\ 0 & 0 & 0 \end{bmatrix}.$$

This shows that if we call the coefficients x, y and z, then if we put $z = 1$, then we can solve for x and y and obtain $y = 2$ and $x = -3$.

$$3\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} - 2\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 5 \end{bmatrix}.$$

b)(7p) Find k so that $w = \begin{bmatrix} 1 \\ -5 \\ k \end{bmatrix}$ is a linear combination of u and v.

Solution

Do the same thing again, we have to get a consistent system.

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & -1 & -5 \\ 3 & 2 & k \end{bmatrix} \xrightarrow{R2 \leftrightarrow R1} \begin{bmatrix} -1 & -1 & -5 \\ 2 & -1 & 1 \\ 3 & 2 & k \end{bmatrix} \xrightarrow{R2+2R1, R3+3R1} \begin{bmatrix} -1 & -1 & -5 \\ 0 & -3 & -9 \\ 0 & -1 & k-15 \end{bmatrix}$$

$$\xrightarrow{R3-R2/3} \begin{bmatrix} \boxed{-1} & -1 & -1 \\ 0 & \boxed{-3} & 6 \\ 0 & 0 & k-12 \end{bmatrix}.$$

Answer: $k = 12$.

Alternative way: find the determinant and set it equal to 0.

3)(14p) Is the matrix

$$A = \begin{bmatrix} 0 & -2 & 4 \\ 1 & 0 & -1 \\ 0 & 4 & -7 \end{bmatrix}$$

invertible? If so, find A^{-1}.

Solution

The determinant is 2, so it is invertible. Find the inverse:

$$[A|I] = \begin{bmatrix} 0 & -2 & 4 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 4 & -7 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -2 & 4 & 1 & 0 & 1 \\ 0 & 4 & -7 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R3+2R2} \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -2 & 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{bmatrix} \xrightarrow{R2-4R3, R1+R3} \begin{bmatrix} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & -2 & 0 & -7 & 0 & -4 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R2/(-2)} \begin{bmatrix} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 7/2 & 0 & 2 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{bmatrix} = [I|A^{-1}].$$

4)(14p) Find the determinant of the matrix

$$A = \begin{bmatrix} -1 & 3 & 2 \\ 4 & -8 & 1 \\ 2 & 2 & 5 \end{bmatrix}.$$

Solution

By the first row it is -(-42)-3(18)+2(24)=42-54+48=36.

5)(14p) Consider the following system in unknowns x and y:

$$\begin{aligned} x - ay &= 1 \\ ax - 4y &= b \end{aligned}$$

For which values of a and b does the system have a unique solution, no solution or infinitely many solutions?

Solution

Subtract a times the first row from the second. We obtain

$$(a^2 - 4)y = b - a.$$

Now if $a^2 \neq 4$, i.e. if $a \neq 2$ or $a \neq -2$, then $y = (b-a)/(a^2-4)$ and $x = 1 + a(b-a)/(a^2-4)$ is the unique solution. If $a = 2$, then the second equation is 2 times the first one (the left hand side), so if $b = 2$, then we have infinitely many solutions, if $b \neq 2$, we have an inconsistent system with no solutions. Same structure for $a = -2$: if $b = -2$, infinitely many solutions, $b \neq -2$, no solutions.

6) Consider the system of ODE's

$$\begin{aligned} x' &= -2y + x(x^2 + 4y^2 - 4) \\ y' &= \tfrac{1}{2}x - y(x^2 + 4y^2 - 4). \end{aligned}$$

a)(6p) Verify that $x = 2\cos t$, $y = \sin t$ is a solution. (See Figure 1. for the phase plane picture of this solution.)

Solution

Plug it in:

$$\begin{aligned} x' &= -2\sin t = -2\sin t + 2\cos t(4\cos^2 t + 4\sin^2 t - 4) \\ y' &= \cos t = \tfrac{1}{2}2\cos t - \sin t(4\cos^2 t + 4\sin^2 t - 4). \end{aligned}$$

b)(8p) Consider the solution to the system with initial conditions $x(0) = 1$, $y(0) = 0$. Is there a time T such that $x(T) = 3$? Explain.

Solution

This is an autonomous equation and the right hand side is nice: continuous, all partials are continuous on the whole plane. So we can use the theorem which states that solution curves will not intersect in the phase plane. The solution for the given IVP starts inside the ellipse, so it can not come out from there. This means there is no T value such that $x(T) = 3$, because the points $(3, y)$ are all outside the ellipse.

7) Consider the initial value problem

$$y' = t^2 \quad y(0) = 0$$

on the interval $[0, 1]$.

a)(5p) Find an exact solution $y = f(t)$ to this problem. Graph this solution over the interval $[0, 1]$.

Solution

$y(t) = t^3/3$. Graph:

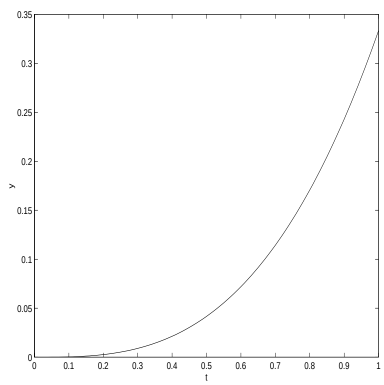

FIGURE 1. $y = t^3/3$

Suppose we apply Euler's method to get a numerical solution:

t	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
y	$y_0 = 0$	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

b)(5p) Compute y_2.

Solution

$t_1 = .1, y_1 = 0 + 0 * .1 = 0$.

$t_2 = .2, y_2 = 0 + .1^2 * .1 = .001$.

c)(6p) Will $y_k > f(t_k) = f(k/10)$ for any $k = 0, \ldots, 10$? Explain.

Solution

No. The reason is that the slope for the exact solution and the computed slope for the approximation is the same at the test points (because the right hand side does not contain y), but in between the slope is strictly increasing for the exact solution and "constant" for the approximation. (If we imagine that we connect the test points and obtain a curve from line pieces.) $y_1 = 0 < f(.1)$, so if we had that somewhere the approximation is bigger than the exact solution, we would have an intersection of the broken-line approximation and the exact solution somewhere between two approximation point. At that point the derivative for the broken-line approximation would be bigger than or equal to the slope of the exact solution, which is a contradiction, it is strictly less there. (Also, the problem can be solved by brute force: by evaluating all the approximating values.)