Math 211

Exam # 1

February 6, 2001

1. Consider the differential equation $x' = \dfrac{t}{x - t^2 x}$.

a) Find the general solution.

Answer: The equation is separable. Separating the variables and integrating we get

$$\begin{aligned} \frac{dx}{dt} &= \frac{t}{x(1-t^2)} \\ \int x\,dx &= \int \frac{t}{1-t^2}\,dt \\ \frac{1}{2}x^2 &= -\frac{1}{2}\ln(|1-t^2|) + C \\ x(t) &= \pm\sqrt{2C - \ln(|1-t^2|)}. \end{aligned}$$

b) Find the solution which satisfies $x(0) = 4$.

Answer: Since $x(0) = 4 > 0$, we have to take the positive square root. Since the initial point is at $t = 0$, where $1 - t^2 = 1 > 0$, we use $|1 - t^2| = 1 - t^2$. Hence $x(t) = \sqrt{2C - \ln(1 - t^2)}$. At $t = 0$ we have $4 = x(0) = \sqrt{2C}$. Hence $C = 8$ and our solution is

$$x(t) = \sqrt{16 - \ln(1 - t^2)}.$$

c) What is the interval of existence for the solution you found in part b)?

Answer: Since $\ln(1 - t^2)$ is only defined if $1 - t^2 > 0$, we must have $|t| < 1$. For t in this range $1 - t^2 \le 1$, so $\ln(1 - t^2) \le 0$. Hence $x(t) = \sqrt{16 - \ln(1 - t^2)}$ is defined. Thus $(-1, 1)$ is the interval of existence.

2. Consider the differential equation $y' = \dfrac{2ty}{1+t^2} + 4t$.

a) Find the general solution.

Answer: The equation is linear. An integrating factor is

$$u(t) = e^{-\int 2t/(1+t^2)\,dt} = e^{-\ln(1+t^2)} = \frac{1}{1+t^2}.$$

Using the solution procedure we have

$$
\begin{aligned}
y' - \frac{2t}{1+t^2}y &= 4t \\
\frac{1}{1+t^2}\left[y' - \frac{2t}{1+t^2}y\right] &= \frac{4t}{1+t^2} \\
\left[\frac{y}{1+t^2}\right]' &= \frac{4t}{1+t^2} \\
\frac{y(t)}{1+t^2} &= \int \frac{4t}{1+t^2}\,dt = 2\ln(1+t^2) + C \\
y(t) &= (1+t^2)[2\ln(1+t^2) + C].
\end{aligned}
$$

b) Find the solution which satisfies $y(0) = 13$.

Answer: We have $13 = y(0) = C$, so the solution is

$$y(t) = (1+t^2)[2\ln(1+t^2) + 13].$$

c) What is the interval of existence for the solution you found in part b)?

Answer: Clearly the solution is defined for all t.

3. Consider a tank which originally contains 100 gallons of pure water. At time $t = 0$ two valves are opened. The first starts a flow of sugar water into the tank at a rate of 1 gallon per minute with a concentration of p lb/gal. The second starts a flow out of the tank at a rate of 2 gallons per minute. As usual we assume perfect and instantaneous mixing.

a) What is the differential equation model for this situation? Include the proper initial condition.

Answer: Let $S(t)$ be the amount of sugar in the tank. Since the tank contains pure water at $t = 0$, $S(0) = 0$. Let $V(t) = 100 - t$ denote the amount of sugar solution in the tank. The rate in is $1\text{gal/min} \times p\text{lb/gal} = p\text{lb/min}$. The rate out is $2\text{gal/min} \times S/V\text{lb/gal} = 2S/V\text{lb/min}$. Hence the model equation is

$$\frac{dS}{dt} = p - \frac{2S}{100-t} \quad \text{with} \quad S(0) = 0.$$

b) Find the general solution for the equation you found in part a).

Answer: The equation is linear. An integrating factor is

$$u(t) = e^{-\int -2/(100-t)\,dt} = e^{-2\ln(100-t)} = \frac{1}{(100-t)^2}.$$

Using the solution procedure we have

$$\begin{aligned} S' + \frac{2S}{100-t} &= p \\ \frac{1}{(100-t)^2}\left[S' + \frac{2S}{100-t}\right] &= \frac{p}{(100-t)^2} \\ \left[\frac{S}{(100-t)^2}\right]' &= \frac{p}{(100-t)^2} \\ \frac{S(t)}{(100-t)^2} &= \int \frac{p}{(100-t)^2}\,dt = \frac{p}{100-t} + C \\ S(t) &= (100-t)[p + C(100-t)]. \end{aligned}$$

c) Find the particular solution that solves the initial value problem you found in part a).

Answer: At $t = 0$ the tank contains pure water, so $0 = S(0) = 100[p + 100C]$. Thus $C = -0.01p$, and the solution is

$$S(t) = p(100-t)[1 - 0.01(100-t)].$$

d) What is p if the concentration of the sugar solution is 1.8 lb/gal when there is 10 gallons left in the tank?

Answer: The concentration is

$$\begin{aligned} c(t) &= \frac{S(t)}{V(t)} \\ &= \frac{p(100-t)[1 - 0.01(100-t)]}{100-t} \\ &= p[1 - 0.01(100-t)] \\ &= 0.01pt. \end{aligned}$$

The volume $V(t) = 100 - t = 10$ when $t = 90$min. At that time the concentration is

$$c(90) = 0.9p$$

If this is equal to 1.8, then $p = 2$.

4. Consider a system of two large tanks, tank A and tank B connected by pipes. Suppose that at time $t = 0$ tank A contains 200 gallons and tank B contains 300 gallons of salt solution at different concentrations. Suppose there is:

- a flow of a salt solution with concentration 2lb/gal into tank A at the rate of 10gal/min
- a flow of the salt solution from tank A to tank B at 10gal/min

- a flow out of tank B at the rate of 5gal/min.

  Let $x(t)$ be the amount of salt in tank A and let $y(t)$ be the amount of salt in tank B at time t.

a) Set up a differential equation that models the rate of change of x.

Answer: The rate in for tank A is 10gal/min $\times$ 2lb/gal $=$ 20lb/min. The rate out is 10gal/min $\times$ $(x/200)$lb/gal $=$ $(x/20)$lb/min. Hence the model equation is

$$\frac{dx}{dt} = 20 - \frac{x}{20}.$$

b) Set up a differential equation that models the rate of change of y.

Answer: The rate in for tank B is the same as the rate out for tank A, $(x/20)$lb/min. The rate out is 5gal/min $\times$ (y/V)lb/gal $=$ $(5y/V)$lb/min, where V is the amount of salt water in tank B. Since $V = 300 + 5t$, the rate out is $y/(60+t)$lb/min. Finally, the model equation is

$$\frac{dy}{dt} = \frac{x}{20} - \frac{y}{60+t}.$$

5. Suppose that y is a solution to the initial value problem

$$y' = \frac{\cos t}{4 - \sin^2 t}(4 - y^2), \quad \text{with} \quad y(0) = 1.$$

Show that

$$\sin t < y(t) < 2, \quad \text{for all } t.$$

Answer: By direct substitution we show that $y_1(t) = \sin t$ and $y_2(t) = 2$ are solutions to the differential equation. The equation has the form $y' = f(t, y)$, where

$$f(t, y) = \frac{\cos t}{4 - \sin^2 t}(4 - y^2).$$

Since $\sin^2 t < 1$ for all t, the function f is continuous on the whole ty-plane. Differentiating we get

$$\frac{\partial f}{\partial y} = \frac{-2y\cos t}{4 - \sin^2 t}.$$

For the same reason this function is also continuous on the whole ty-plane. Thus the hypotheses of the uniqueness theorem are satisfied. We have three solutions y, y_1, and y_2. Since $y(0) = 1$, $y_1(0) = 0$, and $y_2(0) = 2$, we have

$$y_1(0) < y(0) < y_2(0).$$

By the uniqueness theorem the solution curves for the three solutions cannot meet. Hence these inequalities must persist, giving $y_1(t) < y(t) < y_2(t)$ for all t. Thus

$$\sin t < y(t) < 2 \quad \text{for all } t.$$

6. Consider the differential equation

$$y' = 2ty^2 - 4t^4 - 2t^5.$$

a) (5 points) Is the function $y_1(t) = 1 + t^2$ a solution?

Answer: First, for the left hand side of the differential equation we have

$$y_1'(t) = 2t.$$

Next for the right hand side we have

$$\begin{aligned} 2ty_1(t)^2 - 4t^4 - 2t^5 &= 2t(1+t^2)^2 - 4t^4 - 2t^5 \\ &= 2t(1 + 2t^2 + t^4) - 4t^4 - 2t^5 \\ &= 2t. \end{aligned}$$

Since the two sides of the equation are equal, y_1 is a solution.

b) (5 points) Is the function $y_2(t) = t$ a solution?

Answer: For the left hand side of the differential equation we have

$$y_2'(t) = 1.$$

For the right hand side we have

$$2ty_2(t)^2 - 4t^4 - 2t^5 = 2t \cdot t^2 - 4t^4 - 2t^5.$$

Since this is not equal to 1, y_2 is *not* a solution.