

Math 211
First Midterm
February 18, 2003

Make sure to show your work and justify your arguments.

Calculator policy: You may use calculators to evaluate standard functions on floating point numbers (like $\sqrt{3.12}$, $\ln(35/7)$, or $\sin(\pi/17)$). You may not use symbolic operations, numerical integration, or any graphing functions.

1) Suppose that $y(t) = 2e^{-4t}$ is the solution of the initial value problem

$$y' + ky = 0, \quad y(0) = y_0.$$

What are the constants k and y_0 ? (14%)

Solution $y' + ky = -8e^{-4t} + 2ke^{-4t} = 0$, so $k = 4$. Also, $y_0 = y(0) = 2$.

2) Consider the differential equation

$$y' = \frac{1+y}{t^3}.$$

a) Find the general solution. (9%)

b) Find a particular solution with $y(1) = 0$ and identify its interval of existence. (6%)

Solution Separable equation; separate the variables and integrate. We obtain

$$\ln(1+y) = -\frac{1}{2}t^{-2} + C$$

which implies that the general solution is

$$y(t) = Ce^{-\frac{1}{2}t^{-2}} - 1.$$

If $y(1) = 0$ we obtain that $C = e^{1/2}$ and the particular solution is

$$y(t) = e^{-\frac{1}{2}t^{-2} + \frac{1}{2}} - 1.$$

The interval of existence is $(0, \infty)$.

3) Find a solution to the initial value problem

$$2y' + (\cos t)y = -3 \cos t, \quad y(0) = -4. \quad (14\%)$$

Solution This is a linear equation, divide by 2 both sides to get the normal form. The integrating factor is

$$e^{\int \frac{\cos t}{2} dt} = e^{\frac{1}{2} \sin t}.$$

We obtain that

$$(e^{\frac{1}{2} \sin t} y)' = -\frac{3}{2} e^{\frac{1}{2} \sin t} \cos t$$

which gives us

$$e^{\frac{1}{2} \sin t} y = -3e^{\frac{1}{2} \sin t} + C$$

and then

$$y(t) = C e^{-\frac{1}{2} \sin t} - 3.$$

The initial condition gives $C = -1$ and the solution for the IVP is

$$y(t) = -e^{-\frac{1}{2} \sin t} - 3.$$

4) A tank originally contains 100 gal of fresh water. At time $t = 0$, a solution containing 0.2 lb of salt per gallon begins to flow into the tank at a rate of 3 gal/min and the well-stirred mixture flows out of the tank at the same rate.

(a) How much salt is in the tank after 10 min? (6%)

(b) Does the amount of salt approach a limiting value as time increases? If so, what is this limiting value? (6%)

Solution The IVP is

$$S' = .6 - .03S, \quad S(0) = 0.$$

This is a separable equation, after integration we obtain that

$$S(t) = 20(1 - e^{-.03t}).$$

Then $S(10) = 5.1836$ lb. The limit of S as $t \rightarrow \infty$ is 20 lb.

5) A model for population growth is given by the equation

$$P' = rP\left(\frac{P}{\theta} - 1\right)\left(1 - \frac{P}{K}\right),$$

where r, θ and K are given positive constants and $2\theta < K$.

a) Sketch the graph of the function on the right hand side of this differential equation and identify the equilibrium points. (7%)

b) Draw the phase line and analyze the stability near each equilibrium point. (7%)

c) Consider the solution $P(t)$ with initial value $P(0) = K/2$. Describe its behavior as $t \rightarrow \infty$. Does it approach any of the equilibrium solutions? (7%)

Solution The zeroes for the function are 0, θ and K . Also, the leading coefficient of the cubic is a negative number, so the function is negative between 0 and θ and positive between θ and K . This means that 0 and K are stable equilibriums

and θ is unstable. If we start a solution at $K/2 > \theta$, the solution will converge to K in a monotone increasing way.

6) Can you conclude anything about the existence and uniqueness of the solution(s) of the initial value problem

$$y' = \frac{1+y}{t^3}, \quad y(1) = 0? \quad (10\%)$$

Solution The right hand side of the equation and the partial derivative of the right hand side are both continuous in a small enough rectangle around the point $(1, 0)$, so we have a unique solution for the IVP. (What we found in the second problem.)

7) Consider the initial value problem

$$y' = \frac{y-2}{\sin t - 2} \cos t, \quad y(0) = 1.$$

Show that the solution $y(t)$ of this initial value problem satisfies

$$\sin t < y(t) < 2 \text{ for every } t. \quad (14\%)$$

Solution 2 and $\sin t$ are solutions of the differential equation, the initial condition for our solution is between the values of these at the point $t = 0$, so by the uniqueness theorem $\sin t < y(t) < 2$ for all t .