Math 211, Exam 1
September 30th, 2003

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1. (14 points) Find the exact solution of the initial value problem, and indicate the interval of existence.
$$y' = \frac{(y^2+1)}{y}, \quad y(1) = 2.$$

Solution: Separate variables and integrate:
$$\int \frac{y}{y^2+1}\,dy = \int dt \Rightarrow \frac{1}{2}\ln(y^2+1) = t + C.$$
Solving for y, we have
$$\ln(y^2+1) = 2t + C \Rightarrow y^2 + 1 = Ae^{2t}, \quad \text{where } A = e^C$$
$$\Rightarrow y(t) = \pm\sqrt{Ae^{2t}-1}.$$
By virtue of initial value, we note the solution is positive. Therefore,
$$y(t) = \sqrt{Ae^{2t}-1}.$$
It remains to find A:
$$y(1) = 2 \Rightarrow Ae^2 - 1 = 4 \Rightarrow A = 5/e^2,$$
and the solution to the IVP is
$$y(t) = \sqrt{5e^{2t-2}-1}.$$
This solution exists for all values of t such that
$$5e^{2t-2} \geq 1 \;\text{ or }\; e^{2t-2} \geq \frac{1}{5} \;\text{ or }\; 2t - 2 \geq -\ln 5 \;\text{ or }\; 2t \geq -\ln 5 + 2.$$
Therefore, the interval of existence is
$$\left[\frac{-\ln 5}{2} + 1, \;\infty\right), \quad \text{or} \quad \left(\frac{-\ln 5}{2} + 1, \;\infty\right).$$

2. (15 points)

(a) Find the general solution to the differential equation

$$ty' + y = 4t^2.$$

(b) Find the particular solution for the differential equation with initial value $y(1) = 3$. State the interval of existence of the solution.

Solution: (a) Rewrite the equation as

$$y' = -\frac{y}{t} + 4t$$

This is a linear first order equation, and we can do it in two ways:

(I) Using integrating factor: Here, $a(t) = -1/t$, and hence we let $\rho(t) = e^{\int 1/t\ dt} = t$. Then we have $(ty)' = 4t^2$, and integration gives $ty = \frac{4t^3}{3} + C$. Solving for y, we get the general solution

$$y(t) = \frac{4t^2}{3} + \frac{C}{t}.$$

(II) Using variation of parameters: Let y_h be the homogenous solution, i.e. y_h satisfies

$$y_h' = -\frac{y_h}{t} \Rightarrow y_h(t) = \frac{1}{t}.$$

Now, writing $y(t) = v(t)y_h(t)$, for the variable function v, and substituting in to the differential equation, we get

$$v' = \frac{4t}{1/t} = 4t^2.$$

Integrating,

$$v(t) = \frac{4t^3}{3} + C,$$

and this gives

$$y(t) = v(t)y_h(t) = \frac{4t^2}{3} + \frac{C}{t}.$$

(b) We have $y(1) = 3$, thus,

$$\frac{4}{3} + C = 3 \Rightarrow C = 3 - \frac{4}{3} = \frac{5}{3}.$$

Therefore,

$$y(t) = \frac{4t^2}{3} + \frac{5}{3t}.$$

This solution exists for all values of $t \neq 0$, and the interval including the initial value is

$$(0, \infty).$$

3. (14 points) Consider the differential equation $y' = -2t\sqrt{1-y^2}$, where $\sqrt{\ }$ means the positive square root.

(a) Is $y(t) = \sin(t^2)$ a solution?

(b) Is $y(t) = \cos(t^2)$ a solution?

Solution:

(a) For $y(t) = \sin(t^2)$, we get $y' = 2t\cos(t^2)$. On the other hand,

$$-2t\sqrt{1-y^2} = -2t\sqrt{1-\sin^2(t^2)} = -2t\cos(t^2).$$

Clearly, we see that $y' \neq -2t\sqrt{1-y^2}$.

$$\therefore y(t) = \sin(t^2) \text{ is not a solution.}$$

(b) For $y(t) = \cos(t^2)$, we get $y' = -2t\sin(t^2)$. On the other hand,

$$-2t\sqrt{1-y^2} = -2t\sqrt{1-\cos^2(t^2)} = -2t\sin(t^2).$$

Clearly, we see now that $y' = -2t\sqrt{1-y^2}$.

$$\therefore y(t) = \cos(t^2) \text{ is a solution.}$$

4. (14 points) Is it possible to find a function $f(t,x)$ that is continuous and has continuous partial derivatives such that the functions $x_1(t) = t$ and $x_2(t) = \sin t$ are both solutions to $x' = f(t,x)$ near $t = 0$? Why or why not? Explain your reasoning.

Solution: No. Suppose they are both solutions to the differential equation. At $t = 0$, $x_1(0) = x_2(0) = 0$. By uniqueness theorem (since $f(t,x)$ is differentiable and has continuous partial derivative, the uniqueness theorem applies in any rectangle containing (0,0)), $x_1(t)$ and $x_2(t)$ are the same. This is clearly a contradiction.

5. (14 points) A tank contains 100 gal of pure water. A salt solution with concentration 3 lb/gal enters the tank at a rate of 2 gal/min. Solution drains from the tank at a rate of 2 gal/min. Without solving the equation, use qualitative analysis to find the eventual concentration of the salt solution in the tank.

Solution: Let $x(t)$ be the amount of salt in the tank at time t. Then x changes according to the differential equation

$$x' = \underbrace{2\ (\text{gal/min}) \times 3\ (\text{lb/gal})}_{\text{Rate in}} \underbrace{-2\ (\text{gal/min}) \times \frac{x}{100}\ (\text{lb/gal})}_{\text{Rate Out}},$$

or

$$x' = 6 - \frac{2x}{100} \quad (\text{lb/min})$$

We note the equilibrium point of this differential equation is $x = 300$, and is stable. Therefore, the eventual concentration of salt in the tank is

$$\frac{300\ \text{lb}}{100\ \text{gal}} = 3\ \text{lb/gal}.$$

6. (14 points) For the initial value problem

$$y' = ty, \quad y(0) = 1,$$

use Euler's method to compute the first four iterations using step size $h = 1/3$. (i.e., calculate $y_0,\ y_1,\ y_2$, and y_3.)

Solution: With $h = 1/3$, we have $t_0 = 0,\ t_1 = 1/3,\ t_2 = 2/3,\ t_3 = 1$, and use $f(t, y) = ty$.

$$y_0 = y(0) = 1$$

$$y_1 = y_0 + f(t_0, y_0)h = 1 + 0 \cdot 1 \cdot \frac{1}{3} = 1$$

$$y_2 = y_1 + f(t_1, y_1)h = 1 + \frac{1}{3} \cdot 1 \cdot \frac{1}{3} = \frac{10}{9}$$

$$y_3 = y_2 + f(t_2, y_2)h = \frac{10}{9} + \frac{2}{3} \cdot \frac{10}{9} \cdot \frac{1}{3} = \frac{110}{81}.$$

7. (15 points) Consider the autonomous equation

$$y' = y(y-1)(y+2).$$

(a) Find and classify all equilibrium points.

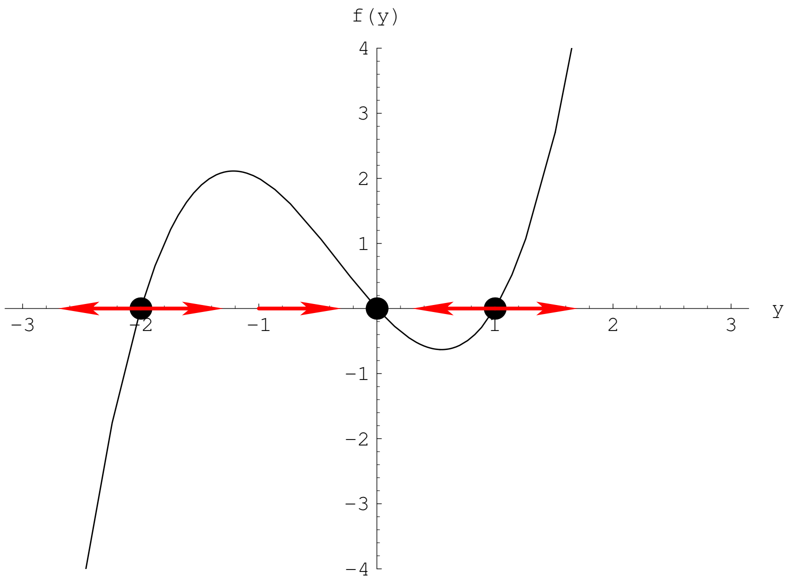
(b) Draw the phase line.

(c) Sketch equilibrium solutions on $ty-$plane. Sketch at least one solution trajectory in each of the regions on $ty-$plane divided by the equilibrium solutions.

Solution:

(a) The RHS $f(y)$ is zero when $y = 0$, $y = 1$ or $y = -2$. These are the equilibrium solutions. Studying the sign of $f(y)$ between equilibrium points, we conclude $y = 0$ is stable, but $y = 1$ and $y = -2$ are unstable equilibrium points.

(b) The phase line is the y-axis below.



(c)