Math 211
First Midterm
September 25, 2001

Make sure to show your work and justify your arguments.
Calculator policy: You may use calculators to evaluate standard functions on floating point numbers (like $\sqrt{3.12}$, $\ln(35/7)$, or $\sin(\pi/17)$). You may not use symbolic operations, numerical integration, or any graphing functions.

1) Consider the differential equation

$$y'' - y' + y = -\cos t.$$

a) Is $y(t) = \sin t$ a solution? (7%)

For this function $y' = \cos t$ and $y'' = -\sin t$, hence

$$\begin{aligned} y'' - y' + y &= \sin t - \cos t - \sin t \\ &= -\cos t. \end{aligned}$$

Therefore, $\sin t$ is a solution.

b) Is $y(t) = 2\sin t$ a solution? (7%)

For this function $y' = 2\cos t$ and $y'' = -2\sin t$, hence

$$\begin{aligned} y'' - y' + y &= 2\sin t - 2\cos t - 2\sin t \\ &= -2\cos t \\ &\neq -\cos t \end{aligned}$$

Therefore, $2\sin t$ is not a solution.

2) Consider the differential equation

$$y'(1-t) = y.$$

a) Give the general solution. (9%)

We use separation of variables:

$$\begin{aligned} (1-t)dy/dt &= y \\ dy/y &= dt/(1-t) \\ \ln|y| &= -\ln|1-t| + C \quad \text{after integration} \\ |y| &= e^C/|1-t| \quad \text{after exponentiation.} \end{aligned}$$

b) Find a particular solution with $y(2) = 1$ and give its interval of existence. (6%)

For these initial conditions $|y| = y$ and $|1-t| = t-1$, so we may write

$$y(t) = e^C/(t-1).$$

Substituting, we find $1 = e^C/1$ and thus

$$y(t) = 1/(t-1)$$

solves the initial value function. This function fails to be defined when $t = 1$, but is well-defined for $t > 1$. The interval of existence is therefore $(1, \infty)$.

3) Find a solution to the initial value problem

$$y' = -\frac{2}{t}y + \frac{\cos t}{t^2} \quad y(\pi) = 0. \quad (14\%)$$

Solution I. (Integrating factor)
Integrating factor: $e^{-\int(-2/t)dt} = e^{2\ln t} = t^2$.
Equation becomes: $t^2y' + 2ty = (t^2y)' = \cos t$.
Then $t^2y = \sin t + C$, $y(\pi) = 0$ gives $C = 0$ and the solution is

$$y(t) = \frac{\sin t}{t^2}.$$

Solution II. (Variation of Parameter)
Homogeneous equation is $y' = -\frac{2}{t}y$, solution for this is
$y(t) = Ct^{-2}$ by separation of variables, then
$y' = C't^{-2} - C2t^{-3} = -2Ct^{-3} + \frac{\cos t}{t^2}$
so $C' = \cos t$ and $C = \sin t + c$,

$$y(t) = (\sin t + c)t^{-2}$$

and $c = 0$ from the initial condition.

4) Consider a pond with 1000 cubic meters of water. There is a stream flowing out from the pond, at a rate of 10 cubic meters a day. Nearby is a field which is regularly irrigated and fertilized. Each day, 10 cubic meters of water from the field enters the pond, and this is contaminated with 3 kilograms of ammonium nitrate per cubic meter.
Write down a differential equation for the amount of ammonium nitrate in the pond. Assume the ammonium nitrate is perfectly mixed and ignore the effect of rain and evaporation. **Do not attempt to solve this differential equation!** (12%)

Let $y(t)$ denote the number of kilograms of ammoniun nitrate in the pond at time t days. We have the balance law

$$\text{rate of change} = \text{amount entering per day} - \text{amount leaving per day}.$$

We compute

$$\text{amount entering} = (10\text{m}^3/\text{day})(3\text{kg.}/\text{m}^3) = 30\text{kg.}/\text{day}$$

and

$$\text{amount leaving} = \frac{10\text{m}^3/\text{day}}{1000\text{m}^3}y(t) = \frac{1}{100}y(t).$$

Hence the differential equation takes the form

$$\frac{dy}{dt} = 30 - y(t)/100.$$

5) Analyze the stability of the differential equation

$$\frac{dy}{dt} = 3 + 2y - y^2.$$

In particular:
a) Sketch the graph of $f(y) = 3 + 2y - y^2$ and identify the equilibrium points. (7%)
The equilibrium points are at $y = -1$ and $y = 3$.

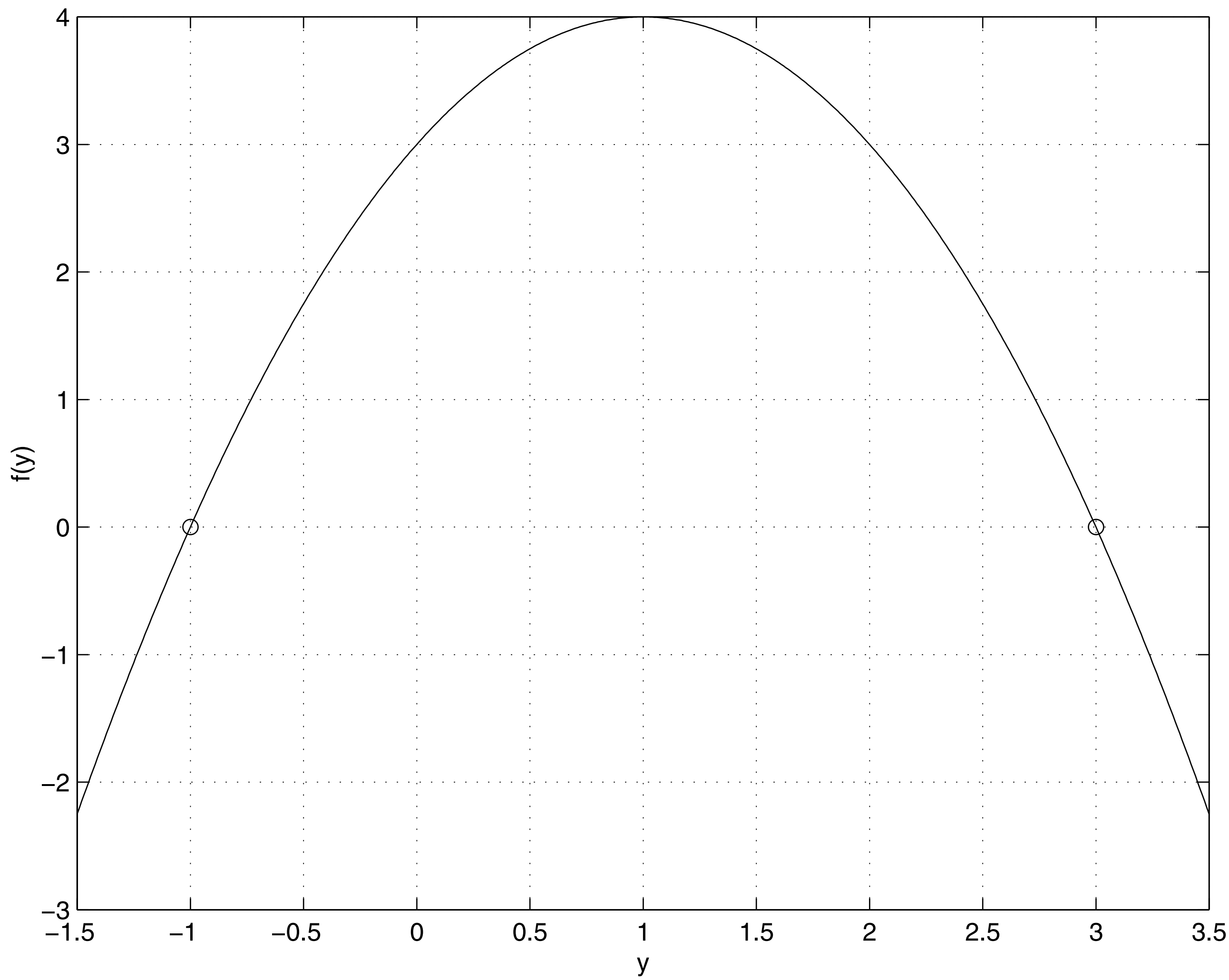

Figure 1: Graph of $f(y)$

b) Draw the phase line and analyze the stability near each equilibrium point. (7%)

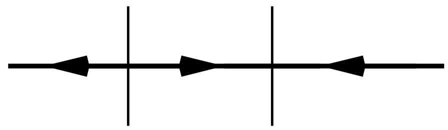

Figure 2: Phase line

The equilibrium $y = 3$ is stable and the equilibrium $y = -1$ is unstable. This can be seen from a qualitative analysis of the phase line, or by observing that $f'(-1) > 0$ and $f'(3) < 0$.

c) Consider the solution $y(t)$ with initial value $y(0) = 0$. Describe its behavior as $t \to \infty$. Does it approach any of the equilibrium solutions? (7%)

We use qualitative analysis. The solution is increasing as a function of t and approaches the stable equilibrium point $y = 3$.

Alternately, we can compute the exact solution to the IVP and show directly that it approaches $y = 3$ as $t \to \infty$. This is *not* required for full credit, but is included here for completeness. Using separation of variables

$$\begin{aligned} -4dt &= \left(\frac{1}{y-3} - \frac{1}{y+1}\right)dy \quad \text{by partial fractions} \\ -4t + C &= \ln|y-3| - \ln|y+1| \quad \text{after integrating} \\ e^{-4t+C} &= \left|\frac{y-3}{y+1}\right| \quad \text{after exponentiating.} \end{aligned}$$

The solution $y(t)$ to the IVP $y(0) = 0$ satisfies

$$-3e^{-4t} = \frac{y-3}{y+1}$$

and solving for y yields

$$y(t) = \frac{3(1-e^{-4t})}{1+e^{-4t}}$$

which approaches 3 as $t \to \infty$.

6) A ball with mass 1/4 kilogram is thrown upward with a high initial velocity. Assume that the air resistance is given by

$$R = -.05v|v|,$$

where v is the velocity, and the gravitational constant is

$$g = 9.8 \text{ meters per second}^2.$$

Compute the terminal velocity of the ball. (10%)

Newton says $ma = -mg + 0.05v^2$ (we will reach terminal velocity coming down, where $v < 0$), reaching terminal velocity means getting close to the asymptotically stable equilibrium $a = 0$, or $0 = -mg + .05v^2$. This means $v_{term} = -\sqrt{mg/.05} = -7$ m per s.

7) Consider the differential equation

$$y' = \frac{e^t(4-y^2)}{4e^t - 2}.$$

a) Show that $y(t) = 2 - 2e^{-t}$ satisfies the differential equation. (7%)
b) Consider the solution $\tilde{y}(t)$ of the initial value problem $y(0) = 1$. Show that

$$\lim_{t\to\infty} \tilde{y}(t) = 2. \quad (7\%)$$

Hint: It is not necessary to compute $\tilde{y}$.

(a) $y = 2 - 2e^{-t}$, so $y' = 2e^{-t}$. Also, $4 - y^2 = 4 - (2 - 2e^{-t})^2 = 8e^{-t} - 4e^{-2t}$ and then

$$\frac{e^t(4-y^2)}{4e^t - 2} = \frac{8 - 4e^t}{4e^t - 2} = 2e^{-t}.$$

(b) The right hand side of the equation satisfies the assumptions of the uniqueness theorem on the half plane $t > \ln(1/2)$, so solutions cannot intersect when $t \geq 0$. Now $y_0(t) = 2 - 2e^{-t}$ is the solution of the IVP with $y_0(0) = 0$ and $y_2(t) = 2$ is the solution of the IVP with $y_2(0) = 2$. The solution of the IVP with $y(0) = 1$ starts between these two, by the uniqueness theorem they cannot intersect, so for all $t > 0$

$$y_0(t) < \tilde{y}(t) < y_2(t).$$

By the squeeze theorem (calc I) we obtain the result.

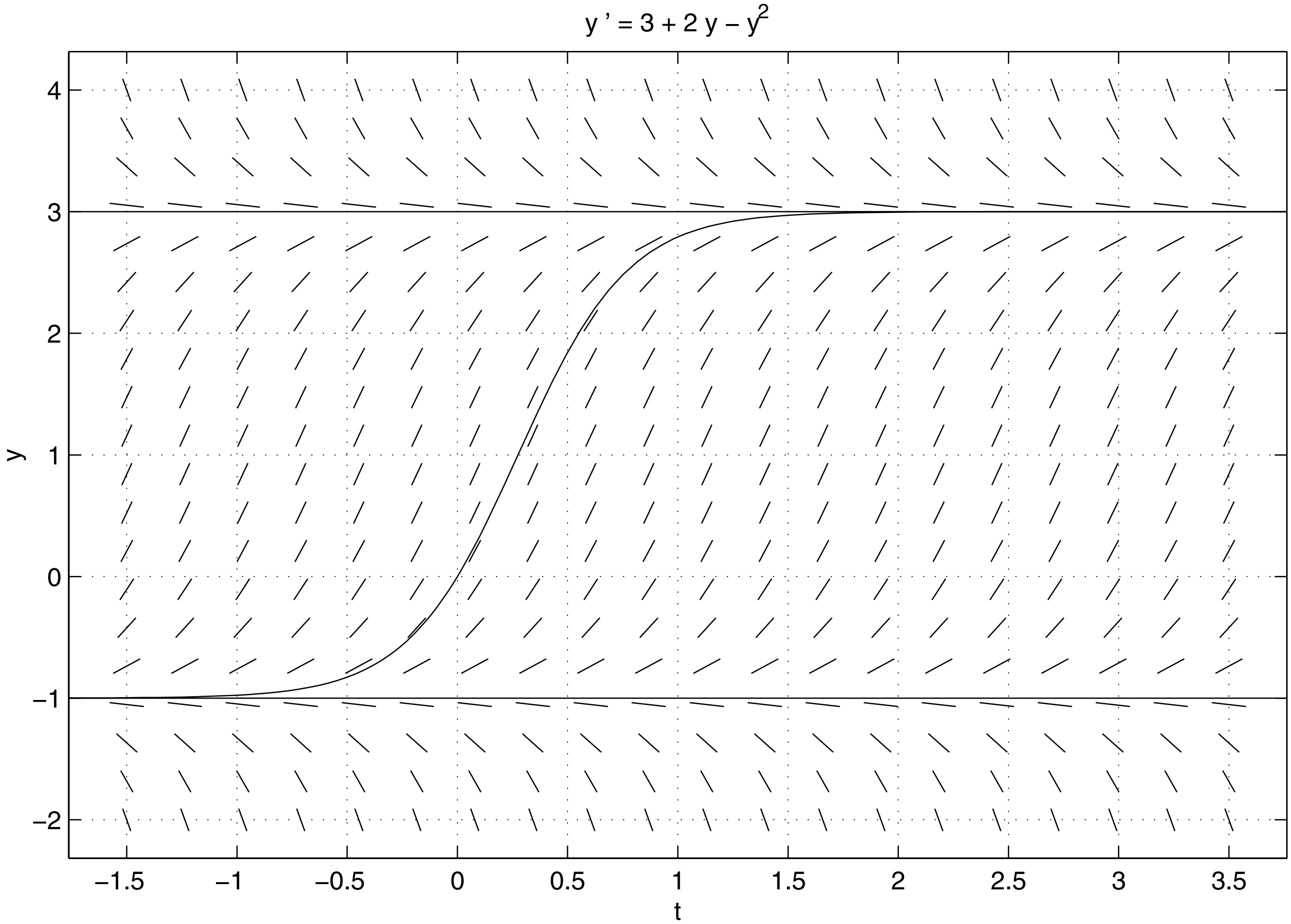

Figure 3: Behavior of solution with $y(0) = 0$