Math 211
First Midterm
September 25, 2001

Make sure to show your work and justify your arguments.

Calculator policy: You may use calculators to evaluate standard functions on floating point numbers (like $\sqrt{3.12}$, $\ln(35/7)$, or $\sin(\pi/17)$). You may not use symbolic operations, numerical integration, or any graphing functions.

1) Consider the differential equation

$$y'' - y' + y = -\cos t.$$

a) Is $y(t) = \sin t$ a solution? (7%)

b) Is $y(t) = 2\sin t$ a solution? (7%)

2) Consider the differential equation

$$y'(1-t) = y.$$

a) Give the general solution. (9%)

b) Find a particular solution with $y(2) = 1$ and give its interval of existence. (6%)

3) Find a solution to the initial value problem

$$y' = -\frac{2}{t}y + \frac{\cos t}{t^2} \quad y(\pi) = 0. \quad (14\%)$$

4) Consider a pond with 1000 cubic meters of water. There is a stream flowing out from the pond, at a rate of 10 cubic meters a day. Nearby is a field which is regularly irrigated and fertilized. Each day, 10 cubic meters of water from the field enters the pond, and this is contaminated with 3 kilograms of ammonium nitrate per cubic meter.

Write down a differential equation for the amount of ammonium nitrate in the pond. Assume the ammonium nitrate is perfectly mixed and ignore the effect of rain and evaporation. **Do not attempt to solve this differential equation!** (12%)

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5) Analyze the stability of the differential equation

$$\frac{dy}{dt} = 3 + 2y - y^2.$$

In particular:

a) Sketch the graph of $f(y) = 3 + 2y - y^2$ and identify the equilibrium points. (7%)

b) Draw the phase line and analyze the stability near each equilibrium point. (7%)

c) Consider the solution $y(t)$ with initial value $y(0) = 0$. Describe its behavior as $t \to \infty$. Does it approach any of the equilibrium solutions? (7%)

6) A ball with mass $1/4$ kilogram is thrown upward with a high initial velocity. Assume that the air resistance is given by

$$R = -.05v|v|,$$

where v is the velocity, and the gravitational constant is

$$g = 9.8 \text{ meters per second}^2.$$

Compute the terminal velocity of the ball. (10%)

7) Consider the differential equation

$$y' = \frac{e^t(4 - y^2)}{4e^t - 2}.$$

a) Show that $y(t) = 2 - 2e^{-t}$ satisfies the differential equation. (7%)

b) Consider the solution $\tilde{y}(t)$ of the initial value problem $y(0) = 1$. Show that

$$\lim_{t \to \infty} \tilde{y}(t) = 2. \quad (7\%)$$

Hint: It is not necessary to compute $\tilde{y}$.