Math 211

Exam # 2

April 11, 2000

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1. Let

$$A = \begin{pmatrix} 0 & 3 & 4 & 2 \\ 1 & 0 & 5 & 3 \\ 1 & 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 4 \\ 16 \\ 11 \end{pmatrix}.$$

a) (10 points) Find the null space of A.

Answer: We use row operations to reduce A to row echelon form. There are many different ways to accomplish this. One way is to

- Interchange rows 1 and 2.
- Add -1 times row 1 from row 3.

  The result is

$$\begin{pmatrix} 1 & 0 & 5 & 3 \\ 0 & 3 & 4 & 2 \\ 0 & 0 & -4 & -1 \end{pmatrix}$$

Thus x_4 is a free variable, so we set $x_4 = t$. Then back-solving yields

$$x_3 = -x_4/4 = -t/4,$$

$$x_2 = (-4x_3 - 2x_4)/3 = (t - 2t)/3 = -t/3, \quad \text{and}$$

$$x_1 = -5x_3 - 3x_4 = 5t/4 - 3t = -7t/4.$$

Thus the nullspace consists of any vector of the form

$$\mathbf{x} = \begin{pmatrix} -7t/4 \\ -t/3 \\ -t/4 \\ t \end{pmatrix} = t \begin{pmatrix} -7/4 \\ -1/3 \\ -1/4 \\ 1 \end{pmatrix},$$

where t is any real number.

b) (10 points) Find the solution space for the system $A\mathbf{x} = \mathbf{b}$.

Answer: We form the augmented matrix

$$M = \begin{pmatrix} 0 & 3 & 4 & 2 & 4 \\ 1 & 0 & 5 & 3 & 16 \\ 1 & 0 & 1 & 2 & 11 \end{pmatrix}.$$

The same row operations used in part a) reduce M to

$$\begin{pmatrix} 1 & 0 & 5 & 3 & 16 \\ 0 & 3 & 4 & 2 & 4 \\ 0 & 0 & -4 & -1 & -5 \end{pmatrix}$$

once more x_4 is free, so we set $x_4 = t$. Then back-solving yields

$$x_3 = (5 - x_4)/4 = (5 - t)/4,$$

$$x_2 = (4 - 4x_3 - 2x_4)/3 = (4 - 5 + t - 2t)/3 = -(1 + t)/3, \quad \text{and}$$
$$x_1 = 16 - 5x_3 - 3x_4 = 16 - 5(5 - t)/4 - 3t = (39 - 7t)/4.$$

Thus the solution space consists of any vector of the form

$$\mathbf{x} = \begin{pmatrix} (39 - 7t)/4 \\ -(1 + t)/3 \\ (5 - t)/4 \\ t \end{pmatrix} = \begin{pmatrix} 39/4 \\ -1/3 \\ 5/4 \\ 0 \end{pmatrix} + t \begin{pmatrix} -7/4 \\ -1/3 \\ -1/4 \\ 1 \end{pmatrix},$$

where t is any real number.

2. Consider the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 8 \\ 6 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_3 = \begin{pmatrix} 4 \\ -24 \\ -34 \end{pmatrix}.$$

a) (10 points) Are the vectors $\mathbf{v}_1,\ \mathbf{v}_2$ and $\mathbf{v}_3$ linearly independent?

Answer: We put the three vectors into a matrix

$$V = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3] = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 8 & -24 \\ -4 & 6 & -34 \end{pmatrix}.$$

We want to find the nullspace, so we use row operations to reduce this to row echelon form. Here is one way to do it.

- Add 4 times row 1 to row 3.
- Multiply row 2 by $1/8$.
- Add -6 times row 2 to row 3. This yields the matrix

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}.$$

We can compute that the determinant of the matrix V is equal to zero from this, so the vectors are linearly dependent. An alternative way to come to this conclusion is to notice that the vector $\mathbf{c} = (4\ -3\ -1)^T$ is in the nullspace of V, so we have

$$4\mathbf{v}_1 - 3\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}.$$

b) (5 points) Find a basis for the span of the vectors $\mathbf{v}_1,\ \mathbf{v}_2,$ and $\mathbf{v}_3$.

Answer: Any vector in the span is of the form

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3.$$

In part a) we noticed that $4\mathbf{v}_1 - 3\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$, or $\mathbf{v}_3 = 4\mathbf{v}_1 - 3\mathbf{v}_2$. Hence $\mathbf{v}$ can be written as

$$\begin{aligned} \mathbf{v} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3(4\mathbf{v}_1 - 3\mathbf{v}_2) \\ &= (c_1 + 4c_3)\mathbf{v}_1 + (c_2 - 3c_3)\mathbf{v}_2. \end{aligned}$$

Thus every vector in the span is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. It is easily seen that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, so $\mathbf{v}_1$ and $\mathbf{v}_2$ are a basis for the span.

In a similar manner it can be shown that any two out of the three vectors form a basis for the span.

c) (10 points) Let

$$\mathbf{v}_4 = \begin{pmatrix} 4 \\ -24 \\ -35 \end{pmatrix}.$$

Are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_4$ linearly independent?

Answer: We proceed in exactly the same way as we did in part a). Now we form

$$V = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_4] = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 8 & -24 \\ -4 & 6 & -35 \end{pmatrix}.$$

The same row operations reduce this matrix to the form

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since this matrix is nonsingular, so is V, and the vectors are linearly independent.

3. Consider the system of differential equations $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

a) (5 points) Show that

$$\mathbf{y}_1(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} \quad \text{and} \quad \mathbf{y}_2(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

are both solutions to the system.

Answer: We compute that

$$\mathbf{y}_1'(t) = \begin{pmatrix} -\sin t \\ -\cos t \end{pmatrix} \quad \text{and} \quad A\mathbf{y}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} = \begin{pmatrix} -\sin t \\ -\cos t \end{pmatrix},$$

so $\mathbf{y}_1$ is a solution. Similarly

$$\mathbf{y}_2' = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} \quad \text{and} \quad A\mathbf{y}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix},$$

so $\mathbf{y}_2$ is also a solution.

b) (8 points) Show that $\mathbf{y}_1$ and $\mathbf{y}_2$ form a fundamental set of solutions.

Answer: We need only show that $\mathbf{y}_1$ and $\mathbf{y}_2$ are linearly independent. We need only check this at the origin. Here we have

$$[\mathbf{y}_1(0)\ \mathbf{y}_2(0)] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

this is the identity matrix, and is clearly nonsingular, so the vectors are linearly independent.

c) (5 points) What is the general solution to the system?

Answer: Since $\mathbf{y}_1$ and $\mathbf{y}_2$ form a fundamental set of solutions the general solution is the general linear combination of the two. Hence

$$\mathbf{y}(t) = C_1\mathbf{y}_1(t) + C_2\mathbf{y}_2(t) = C_1\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + C_2\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

d) (7 points) Find the particular solution $\mathbf{y}(t)$ to the system which satisfies the initial condition

$$\mathbf{y}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Answer: We need to find the constants C_1 and C_2 such that

$$\mathbf{y}(0) = C_1\mathbf{y}_1(0) + C_2\mathbf{y}_2(0) = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

clearly we want $C_1 = 1$ and $C_2 = -1$, and the solution is

$$\mathbf{y}(t) = \mathbf{y}_1(t) - \mathbf{y}_2(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} - \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} = \begin{pmatrix} \cos t - \sin t \\ -\sin t - \cos t \end{pmatrix}.$$

4. Let

$$A = \begin{pmatrix} -3 & -1 \\ 2 & 0 \end{pmatrix}.$$

a) (5 points) Show that A has eigenvalues -1 and -2.

Answer: We compute the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I) = (-3-\lambda)(-\lambda) + 2 = \lambda^2 + 3\lambda + 2.$$

By using either the factorization $p(\lambda) = (\lambda+2)(\lambda+1)$ or the quadratic formula we conclude that the roots of p are -1 and -2. These are the eigenvalues.

b) (10 points) For each of these eigenvalues find an eigenvector.

Answer: For the eigenvalue $\lambda_1 = -1$ we have

$$A - \lambda_1 I = A + I = \begin{pmatrix} -2 & -1 \\ 2 & 1 \end{pmatrix}.$$

The vector

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

is in the nullspace and hence is an eigenvector.

For the eigenvalue $\lambda_2 = -2$ we have

$$A - \lambda_2 I = A + 2I = \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}.$$

The vector

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is in the nullspace and hence is an eigenvector.

c) (10 points) Find a fundamental set of solutions for the system $\mathbf{y}' = A\mathbf{y}$.

Answer: We set

$$\mathbf{y}_1(t) = e^{\lambda_1 t}\mathbf{v}_1 = e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \mathbf{y}_2(t) = e^{\lambda_2 t}\mathbf{v}_2 = e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Since the eigenvalues λ_1 and λ_2 are distinct, these are linearly independent and therefore form a fundamental set of solutions.

5. (5 points) Compute the determinant of

$$A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ -1 & 0 & 1 & 0 \\ 0 & 3 & 4 & 5 \\ -3 & 0 & 0 & -2 \end{pmatrix}.$$

Answer: There are many ways to compute this determinant using row operations and expansion along rows and columns. Here is one way.

Use the row operations

- Add row 1 to row 2.
- Add 3 times row 1 to row 4.

These do not change the determinant, so

$$\det(A) = \det \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 2 & 1 & 4 \\ 0 & 3 & 4 & 5 \\ 0 & 6 & 0 & 10 \end{pmatrix}.$$

Next we expand along the first column to get

$$\det(A) = \det \begin{pmatrix} 2 & 1 & 4 \\ 3 & 4 & 5 \\ 6 & 0 & 10 \end{pmatrix}.$$

Expanding along the third row we get

$$\begin{aligned} \det(A) &= 6 \det \begin{pmatrix} 1 & 4 \\ 4 & 5 \end{pmatrix} + 10 \det \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \\ &= 6 \times (-11) + 10 \times 5 \\ &= -16. \end{aligned}$$