Math 211

Exam # 2

November 9, 1999

Part 2

Instructions: Part 2 of Exam #2 is an open book, open notes, untimed, take home exam. It is due in the Mathematics Department Office by 4:00 PM on Friday, November 12.

Notice that the points on the questions add to 40.

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1. (12 points) Consider the matrix

$$A = \begin{pmatrix} -1 & 2 & 3 & -4 & 1 \\ 69 & -98 & -7 & 86 & -19 \\ 10 & -12 & -2 & 11 & -3 \\ 70 & -100 & -10 & 90 & -20 \\ -56 & 78 & 4 & -67 & 15 \end{pmatrix}.$$

Find the solution space of the systems of equations $A\mathbf{x} = \mathbf{0}$, $A\mathbf{x} = \mathbf{b}_1$, and $A\mathbf{x} = \mathbf{b}_2$, where

$$\mathbf{b}_1 = \begin{pmatrix} 5 \\ 115 \\ 12 \\ 110 \\ -96 \end{pmatrix} \quad \text{and} \quad \mathbf{b}_2 = \begin{pmatrix} 5 \\ 115 \\ 11 \\ 110 \\ -96 \end{pmatrix}.$$

You are allowed to use any technology available to you.

Answer: MATLAB gives

```
>> null(A,'r')

ans =

    0.2500    0.2500
    1.0000         0
    0.7500   -0.2500
    1.0000         0
         0    1.0000
```

We can simplify the form of these vectors by mulitpying them by 4. Hence the nullspace of A, or, what is the same thing, the solution space of $A\mathbf{y} = \mathbf{0}$, is the set of all vectors of the form

$$s\mathbf{v}_1 + t\mathbf{v}_2, \quad \text{where} \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 4 \\ 3 \\ 4 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 4 \end{pmatrix}.$$

This problem can also be done using `rref` as is done for the other two right hand sides.

For the right hand side $\mathbf{b}_1$, we get for the augmented matrix

```
>> rref([A,b1])

ans =

    1.0000         0         0   -0.2500   -0.2500    0.5000
         0    1.0000         0   -1.0000         0   -1.0000
         0         0    1.0000   -0.7500    0.2500    2.5000
         0         0         0         0         0         0
         0         0         0         0         0         0
```

If we let $x_4 = s$ and $x_5 = t$ be free, then we get

$$\begin{aligned} x_1 &= 1/2 + x_4/4 + x_5/4 = 1/2 + s/4 + t/4 \\ x_2 &= -1 + x_4 = -1 + s \\ x_3 &= 5/2 + 3x_4/4 - x_5/4 = 5/2 + 3s/4 - t/4 \end{aligned}$$

Thus the vectors in the solution space are those of the form

$$\mathbf{x} = \begin{pmatrix} 1/2 + s/4 + t/4 \\ -1 + s \\ 5/2 + 3s/4 - t4t \\ s \\ t \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1 \\ 5/2 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1/4 \\ 1 \\ 3/4 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1/4 \\ 0 \\ -1/4 \\ 0 \\ 1 \end{pmatrix}.$$

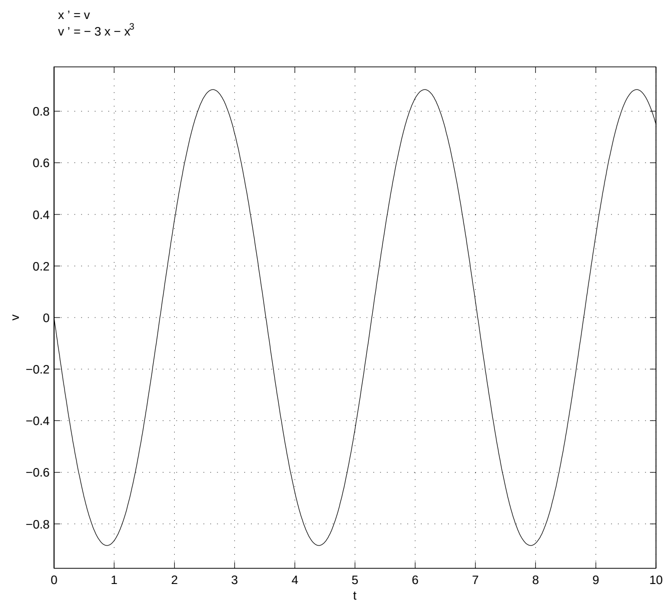
Finally for the right hand side $\mathbf{b}_2$, we have

```
>> rref([A,b2])

ans =

    1.0000         0         0   -0.2500   -0.2500         0
         0    1.0000         0   -1.0000         0         0
         0         0    1.0000   -0.7500    0.2500         0
         0         0         0         0         0    1.0000
         0         0         0         0         0         0
```

The fourth equation is $0 = 1$. We conclude that the system is inconsistent, so there are no solutions. The solution space is the empty set.

2. For a linear spring the restoring force is $R = -kx$, where x is the displacement from equilibrium, and k is the spring constant. For a linear spring without damping the frequency of oscillation is always the natural frequency $\omega_0 = \sqrt{k/m}$, where m is the mass. The period of this oscillation is $T = 2\pi/\omega_0$. The frequency and period do not depend on the amplitude of the oscillation.

A **hard** spring is one where the restoring force increases in magnitude faster than the linear relationship of the linear spring. For example, we could have $R = -kx - lx^3$, where l is a positive constant. Suppose we have a hard spring with restoring force $R = -3x - x^3$. The mass is $m = 1$, and there is no damping. Discover whether the period of oscillation of this spring varies as the amplitude increases. To be precise, use `pplane5` to measure the period of oscillation for amplitudes of 0.1 through 1, with an increment of 0.1. For each amplitude use `pplane5` to compute and plot the solution with initial displacement equal to that amplitude and initial velocity equal to 0. Then use the graphical representations available in `pplane5` to estimate the period. You will have to collect the data by hand and then plot it. If you put the data into two vectors, you can use MATLAB to construct the plot.



The plot of velocity vs. time for the amplitude of 0.5.

a) (2 points) What is the second order differential equation that models this hard spring?

Answer:

$$x'' + 3x + x^3 = 0.$$

b) (2 points) What is the equivalent first order system?

Answer:

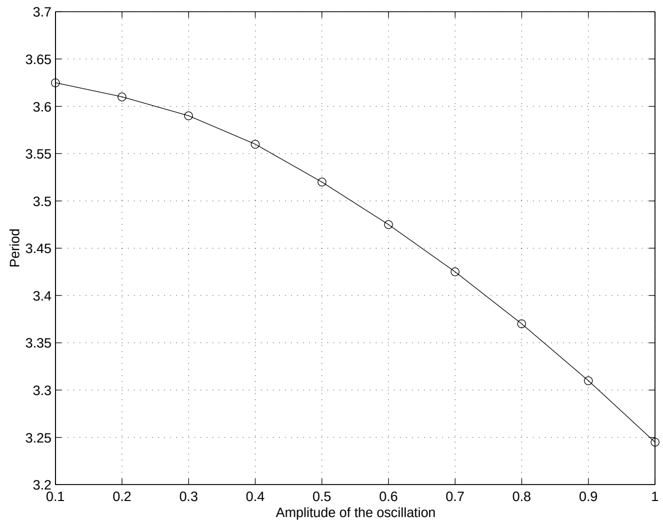
$$x' = v$$
$$v' = -3x - x^3$$

c) (2 points) Submit the `pplane5` plot that you used to determine the period for one of the amplitudes.

Answer: For the amplitude of 0.5 we have the graph on the preceding page. Two periods took a time of about 7, so the period is about 3.5.

d) (4 points) Submit a plot of period versus amplitude.

Answer:



Variation of the period of a hard spring with the amplitude of the oscillation.

e) (4 points) Write a short paragraph summarizing your results, emphasizing how the motion of the nonlinear spring differs from the linear spring.

Answer: The period of the oscillation decreases as the amplitude increases. This is unlike the linear case where the period does not depend on the amplitude of the oscillation.

Hints:

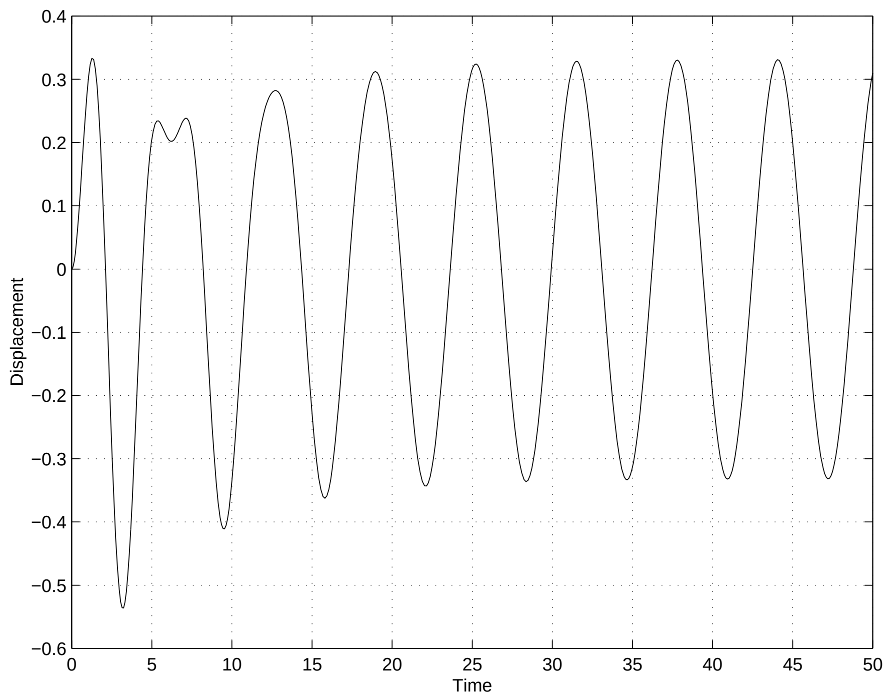
- You can modify the vibrating spring system in the `pplane5` Gallery menu to model the hard spring.
- It would be best to use Keyboard Input to enter your amplitude with zero velocity. Specify a computation interval covering 2 or 3 periods.
- It is impossible to measure the period using the phase plane, so use the Graph menu to get other representations. With these you can measure the time it takes to complete a period.
- You can use the graph of the displacement, but then you have to estimate where a maximum occurs, which is difficult. On the other hand the velocity is equal to 0 when the displacement is at a maximum (or a minimum). It is easier to estimate the location of a zero crossing than to estimate the location of a maximum.
- If you measure the time it takes to have 2 or 3 periods and then divide by the number of periods you will increase accuracy.

- Using the crop feature will enable you to be more accurate in estimating the period. (The crop feature is explained in the Manual.)

3. Consider a damped spring that is being forced by a sinusoidal force. It is modeled by the differential equation

$$mx'' + \mu x' + kx = A\cos(\omega t),$$

where m is the mass, μ is the damping constant, k is the spring constant. The forcing term has amplitude A and driving frequency ω. Such a spring has a complicated oscillation, which in a short time settles into a steady-state, sinusoidal form. An example is shown in the figure below. In this problem you are to discover with a computational experiment how the amplitude of the steady-state oscillation depends on the driving frequency.



The oscillation of a forced, damped spring.

To be specific, assume that $m = 1$, $\mu = 0.3$, $k = 4$, and $A = 1$. For $\omega = 0.5,\ 1,\ 1.5,\ 2,\ 2.5,\ 3,\ 3.5,$ and 4, use `ode45` to compute the solution with the spring starting at rest and plot the displacement versus time. From this plot estimate the amplitude of the steady-state oscillation. You will have to collect the data by hand and then plot it.

a) (4 points) Submit the m-file that you used to solve the equation.

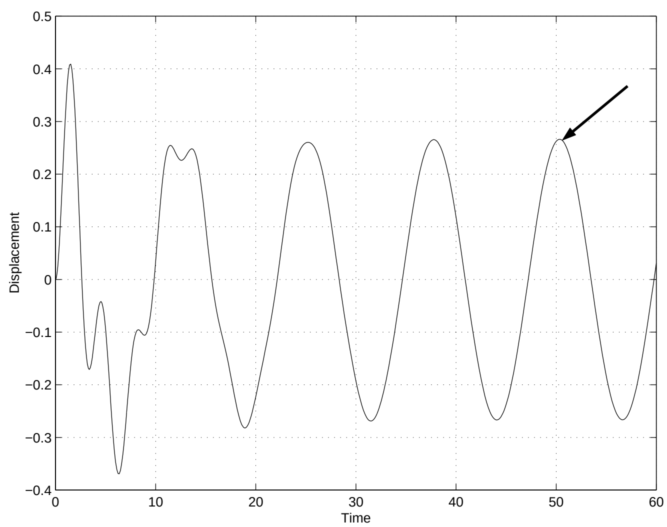
Answer: There is a large number of ways to write the derivative m-file. Here is only one possibility.

```
function upr = fdlsp(t,u)

  global OM
  x = u(1); v = u(2);

  xpr = v;
  vpr = -4*x - 0.3*v +cos(OM*t);
  upr = [xpr;vpr];
```

This way of writing the file makes the file easier to understand, but it is not the shortest possible. It enters the frequency as the global variable `OM`, to allow it to be easily changed.

b) (4 points) For one of the driving frequencies, submit the plot of displacement versus time over the full time period of your computation. Indicate on this plot the point that you used to estimate the amplitude.

Answer: The following graph plots displacement vs. time for a driving frequency of $\omega = 0.5$. The arrow shows the point at which the amplitude was measured.



Displacement vs. time for $\omega = 0.5$.

c) (6 points) Submit a plot of amplitude versus driving frequency. For which frequency is the amplitude largest?

Answer: The graph on the next page shows the data collected. There is a sharp peak in the amplitude at $\omega = 2$. (It is not necessary for the students to point out that this is the natural frequency of the spring. However, it is very interesting.)

Hints:

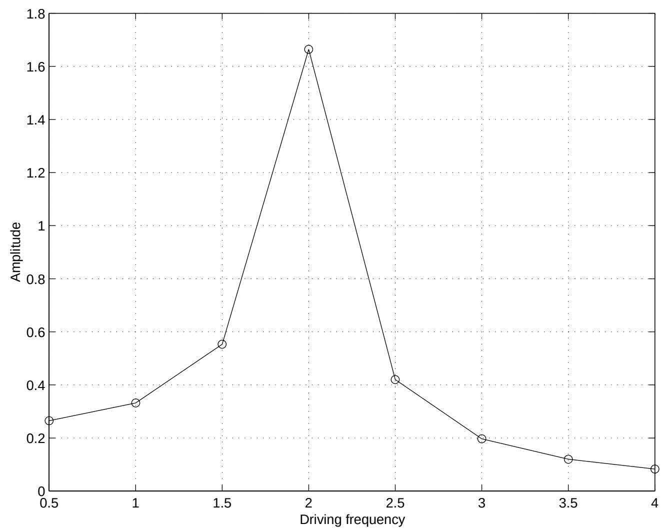
- Be sure you have computed over a sufficiently long interval to obtain the steady-state oscillation.

- Put a grid on your plot to help you estimate the amplitude.

- Use the zoom feature of the MATLAB graph. To do so, click on the magnifying glass in the Toolbar with a plus sign in it. When this button is highlighted, and you click the left button of the mouse on a maximum, the figure will zoom in. You can do this several times to get better accuracy. If necessary, you can zoom out using the other button on the Toolbar (or by using the right mouse button).

  If you are using a Macintosh there is no Toolbar. Instead you do a help on zoom, and follow the intructions.

- It turns out that the steady-state oscillation does not depend on the intitial conditions, so starting at rest is as good as any other starting point for this experiment.



Amplitude vs. driving frequency ω.