Math 211

Exam # 2

November 9, 1999

Part 2

Instructions: Part 2 of Exam #2 is an open book, open notes, untimed, take home exam. It is due in the Mathematics Department Office by 4:00 PM on Friday, November 12. You may not discuss the exam with your fellow students. If you have any questions, consult one of the instructors for the course.

Write out and sign the pledge on your submitted solutions. Be sure to put your name on your submission. In addition put the name of your instructor at the top of the first page of your submission.

Please give reasons for all of your answers. Remember that some reasons are better than others. For example, it is better to refer to a theorem stated in class and/or in the books than to say "the computer printout shows that ...". Of course sometimes the computer printout is all you have.

In answering these questions you are not limited to the use of MATLAB, although it will be useful. You should use any combination of the analytic, qualitative, or numerical methods you have learned.

Your answers should consist of complete sentences organized into coherent paragraphs. This does not mean that they have to be long. Brevity is frequently a sign of understanding. For much of this exam you can record what you are doing in a MATLAB diary file, and use that as a start of your exam submission. MATLAB graphics should be numbered and referred to by that number. Any graphic which is not referred to will not be counted as part of your submission.

This exam contains large matrices and vectors. To eliminate mistakes that are inevitable when copying these to your computer we have automated the process. If you are working on an owlnet computer, you need only execute `exam2` at the MATLAB prompt and all of the vectors and matrices will entered for you. If you are working outside of owlnet, go to the Math 211 homepage and to the description of the second exam. There is a link there to **exam2.m**. Click on the link and you will see the file. Select and copy all of the commands, and paste them to the MATLAB Command Window. Hit Enter, and all of the data will be in the MATLAB workspace. Execute the MATLAB command `who` to be sure.

Notice that the points on the questions add to 40.

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1. (12 points) Consider the matrix

$$A = \begin{pmatrix} -1 & 2 & 3 & -4 & 1 \\ 69 & -98 & -7 & 86 & -19 \\ 10 & -12 & -2 & 11 & -3 \\ 70 & -100 & -10 & 90 & -20 \\ -56 & 78 & 4 & -67 & 15 \end{pmatrix}.$$

Find the solution space of the systems of equations $A\mathbf{x} = \mathbf{0}$, $A\mathbf{x} = \mathbf{b}_1$, and $A\mathbf{x} = \mathbf{b}_2$, where

$$\mathbf{b}_1 = \begin{pmatrix} 5 \\ 115 \\ 12 \\ 110 \\ -96 \end{pmatrix} \quad \text{and} \quad \mathbf{b}_2 = \begin{pmatrix} 5 \\ 115 \\ 11 \\ 110 \\ -96 \end{pmatrix}.$$

You are allowed to use any technology available to you.

2. For a linear spring the restoring force is $R = -kx$, where x is the displacement from equilibrium, and k is the spring constant. For a linear spring without damping the frequency of oscillation is always the natural frequency $\omega_0 = \sqrt{k/m}$, where m is the mass. The period of this oscillation is $T = 2\pi/\omega_0$. The frequency and period do not depend on the amplitude of the oscillation.

A **hard** spring is one where the restoring force increases in magnitude faster than the linear relationship of the linear spring. For example, we could have $R = -kx - lx^3$, where l is a positive constant. Suppose we have a hard spring with restoring force $R = -3x - x^3$. The mass is $m = 1$, and there is no damping. Discover whether the period of oscillation of this spring varies as the amplitude increases. To be precise, use `pplane5` to measure the period of oscillation for amplitudes of 0.1 through 1, with an increment of 0.1. For each amplitude use `pplane5` to compute and plot the solution with initial displacement equal to that amplitude and initial velocity equal to 0. Then use the graphical representations available in `pplane5` to estimate the period. You will have to collect the data by hand and then plot it. If you put the data into two vectors, you can use MATLAB to construct the plot.

a) (2 points) What is the second order differential equation that models this hard spring?

b) (2 points) What is the equivalent first order system?

c) (2 points) Submit the `pplane5` plot that you used to determine the period for one of the amplitudes.

d) (4 points) Submit a plot of period versus amplitude.

e) (4 points) Write a short paragraph summarizing your results, emphasizing how the motion of the nonlinear spring differs from the linear spring.

Hints:

- You can modify the vibrating spring system in the `pplane5` Gallery menu to model the hard spring.
- It would be best to use Keyboard Input to enter your amplitude with zero velocity. Specify a computation interval covering 2 or 3 periods.
- It is impossible to measure the period using the phase plane, so use the Graph menu to get other representations. With these you can measure the time it takes to complete a period.
- You can use the graph of the displacement, but then you have to estimate where a maximum occurs, which is difficult. On the other hand the velocity is equal to 0 when the displacement is at

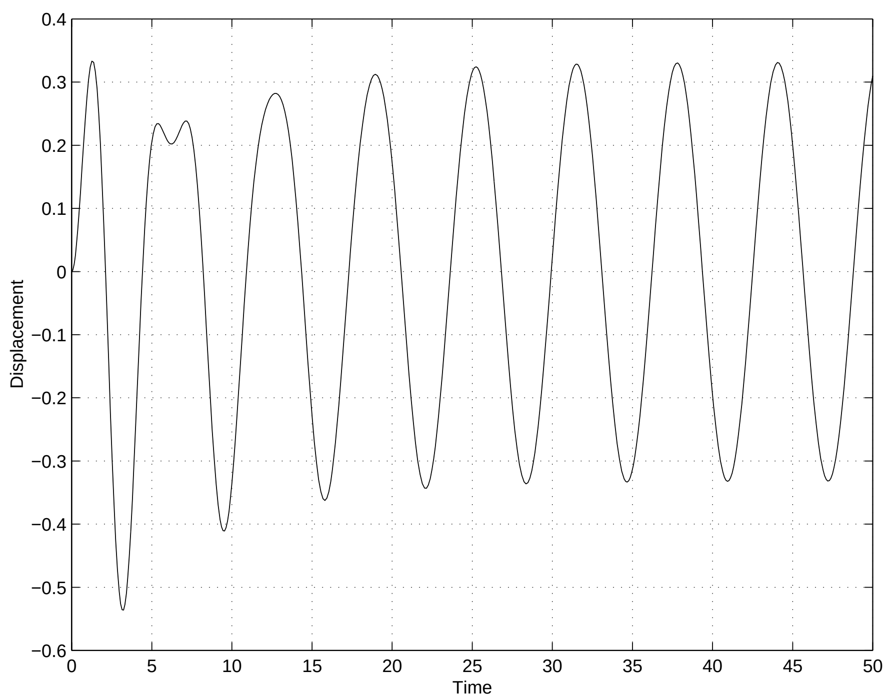
a maximum (or a minimum). It is easier to estimate the location of a zero crossing than to estimate the location of a maximum.

- If you measure the time it takes to have 2 or 3 periods and then divide by the number of periods you will increase accuracy.
- Using the crop feature will enable you to be more accurate in estimating the period. (The crop feature is explained in the Manual.)

3. Consider a damped spring that is being forced by a sinusoidal force. It is modeled by the differential equation

$$mx'' + \mu x' + kx = A\cos(\omega t),$$

where m is the mass, μ is the damping constant, k is the spring constant. The forcing term has amplitude A and driving frequency ω. Such a spring has a complicated oscillation, which in a short time settles into a steady-state, sinusoidal form. An example is shown in the figure below. In this problem you are to discover with a computational experiment how the amplitude of the steady-state oscillation depends on the driving frequency.



The oscillation of a forced, damped spring.

To be specific, assume that $m = 1$, $\mu = 0.3$, $k = 4$, and $A = 1$. For $\omega = 0.5,\ 1,\ 1.5,\ 2,\ 2.5,\ 3,\ 3.5$, and 4, use `ode45` to compute the solution with the spring starting at rest and plot the displacement versus time. From this plot estimate the amplitude of the steady-state oscillation. You will have to collect the data by hand and then plot it.

a) (4 points) Submit the m-file that you used to solve the equation.

b) (4 points) For one of the driving frequencies, submit the plot of displacement versus time over the full time period of your computation. Indicate on this plot the point that you used to estimate the amplitude.

c) (6 points) Submit a plot of amplitude versus driving frequency. For which frequency is the amplitude largest?

Hints:

- Be sure you have computed over a sufficiently long interval to obtain the steady-state oscillation.

- Put a grid on your plot to help you estimate the amplitude.

- Use the zoom feature of the MATLAB graph. To do so, click on the magnifying glass in the Toolbar with a plus sign in it. When this button is highlighted, and you click the left button of the mouse on a maximum, the figure will zoom in. You can do this several times to get better accuracy. If necessary, you can zoom out using the other button on the Toolbar (or by using the right mouse button).

  If you are using a Macintosh there is no Toolbar. Instead you do a help on zoom, and follow the intructions.

- It turns out that the steady-state oscillation does not depend on the intitial conditions, so starting at rest is as good as any other starting point for this experiment.