Math 211

Exam # 1

February 29, 2000

1. (15 points) Consider the differential equation $x' = 3t^3(1 + x^2)$.

a) Find the general solution.

Answer: This equation is separable. We separate variables in

$$\frac{dx}{dt} = 3t^2(1 + x^2)$$

to get

$$\frac{dx}{1 + x^2} = 3t^2\,dt.$$

Integrating we get

$$\arctan(x) = t^3 + C.$$

Solving for x we find the general solution

$$x(t) = \tan(t^3 + C).$$

b) Find the solution which satisfies $x(0) = 0$.

Answer: The initial condition gives

$$0 = x(0) = \tan(C).$$

Hence $C = 0$, and the solution to the initial value problem is

$$x(t) = \tan(t^3).$$

c) What is the interval of existence for the solution you found in part b)?

Answer: In order for $\tan(t^3)$ to be defined we need $t^3 \in (-\pi/2, \pi/2)$. Hence the interval of existence is

$$-\left(\frac{\pi}{2}\right)^{1/3} < t < \left(\frac{\pi}{2}\right)^{1/3}.$$

2. (15 points) Consider the differential equation $y' = \cos(t) - y\tan(t)$.

a) Find the general solution.

Answer: This equation is linear. The coefficient of y is $a(t) = -\tan(t)$. An integrating factor is given by

$$u(t) = e^{-\int a(t)\,dt}.$$

We compute

$$-\int a(t)\,dt = \int \tan(t)\,dt = \int \frac{\sin(t)\,dt}{\cos(t)} = -\ln(\cos(t)),$$

and

$$u(t) = e^{-\ln(\cos(t))} = \frac{1}{\cos(t)}.$$

Rewriting the differential equation as

$$y' + y\tan(t) = \cos(t),$$

and multiplying by the integrating factor, we get

$$\begin{aligned}\frac{1}{\cos(t)}(y' + y\tan(t)) &= \frac{y'}{\cos(t)} + \frac{y\ \sin(t)}{\cos^2(t)} \\ &= \left(\frac{y}{\cos(t)}\right)' \\ &= 1.\end{aligned}$$

Integrating we get

$$\frac{y}{\cos(t)} = t + C.$$

Hence the general solution is

$$y(t) = (t + C)\cos(t).$$

b) Find the solution which satisfies $y(0) = 1$.

Answer: The initial condition gives us

$$1 = y(0) = C.$$

Hence our solution is

$$y(t) = (t + 1)\cos(t).$$

c) What is the interval of existence for the solution you found in part b)?

Answer: Clearly the solution is defined over the whole real line. However, because of $\tan t$ is a coefficient of the original equation, the equation itself is not defined when t is an odd multiple of $\pi/2$. The correct answer is therefore the interval $(-\pi/2, \pi/2)$. Both answers were marked correct.

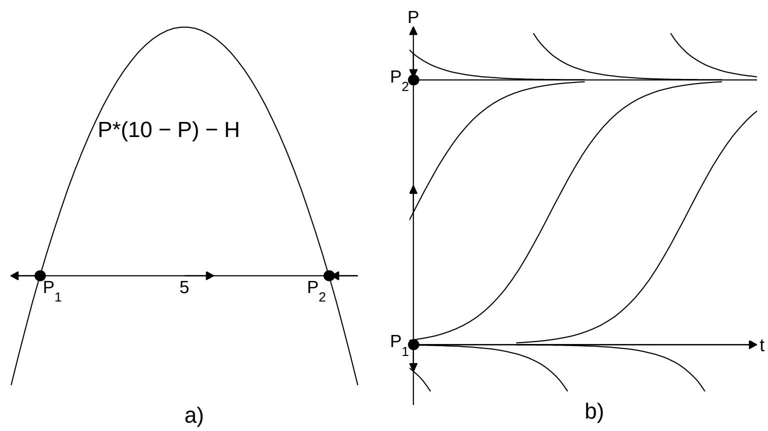
3. (18 points) Suppose $P(t)$ is the fish population (in millions) in Lake Houston at time t. We model this population with the differential equation

$$\frac{dP}{dt} = P(10 - P) - H$$

where H is a constant depending on the amount of fishing occurring in our region.

Answer: The equilibrium points are the solutions to $P(10 - P) - H = 0$, or $P^2 - 10P + H = 0$. Assuming that $H < 25$, the quadratic formula yields two roots

$$P_1 = 5 - \sqrt{25 - H} \quad \text{and} \quad P_2 = 5 + \sqrt{25 - H}.$$



The general situation is shown in the above figure. Figure a) shows the plot of $P(10-P)-H$. The x-axis shows a phase line. Figure b) shows the plot of a few curves, including the equilibrium solutions $P(t) = P_1$ and $P(t) = P_2$. In this case the P-axis is the phase line. It is also possible to display a phase line which is not an axis.

a) Draw the phase line for the case that no fishing occurs (i.e. $H = 0$). Sketch a few solutions, including the equilibrium solutions.

Answer: In this case $P_1 = 0$ and $P_2 = 10$. The picture applies.

b) Draw the phase line for the case $H = 16$. Sketch a few solutions, including the equilibrium solutions.

Answer: In this case $P_1 = 2$ and $P_2 = 8$. The picture applies.

c) Suppose the fish population has declined to $P(t_0) = 1$ at some time t_0. What values of H will ensure that the fish population won't go to zero?

Answer: It is necessary that $P(t_0) = 1$ lies between the two equilibrium points, or $P_1 < 1 < P_2$. Since $P_2 > 5$, the only constraint is that $P_1 = 5 - \sqrt{25 - H} < 1$ or $\sqrt{25 - H} > 4$. This will be true if $25 - H > 16$ or $H < 9$.

4. (10 points) Consider the following system, consisting of two ponds, A and B, each of which contains a certain number of parasites.

- The volume of pond A is 1000 liters, and the volume of pond B is 500 liters.
- Parasites breed at a rate proportional to the number of parasites in the same pond.
- Clean water is flowing into pond A at 20 liters per minute.
- Well-mixed water flows from pond A to pond B at 20 liters per minute.
- Well-mixed water flows out of pond B at 20 liters per minute.

Set up a system of two differential equations modeling the numbers of parasites in ponds A and B at any time t. **You are not required to solve the system of differential equations.**

Answer: For each of the tanks we use the basic equation

$$\text{rate of change } = \text{rate in} - \text{rate out.}$$

Let $x(t)$ denote the number of parasites in tank A, and $y(t)$ the number in tank B. Parasites enter tank A only through breeding, so the rate in is proportional to x. Therefore there is a constant a such that the rate in is equal to ax. Parasites flow out of tank A with the water. The rate at which they flow out is equal to the rate of flow of the water times the concentration of the parasites. Hence the rate out is equal to $20 \times x/1000$. Therefore the differential equation for x is

$$\frac{dx}{dt} = ax - \frac{x}{50}.$$

Parasites enter tank B from tank A at the same rate that they leave tank A, or $x/50$. New parasites are bred at a rate of ay. Parasites flow out of tank B at a rate equal to the rate of flow of the water times the concentration of the parasites, or $20 \times y/500$. Therefore the differential equation for y is

$$\frac{dy}{dt} = ay + \frac{x}{50} - \frac{y}{25}.$$

Thus the system of equations is

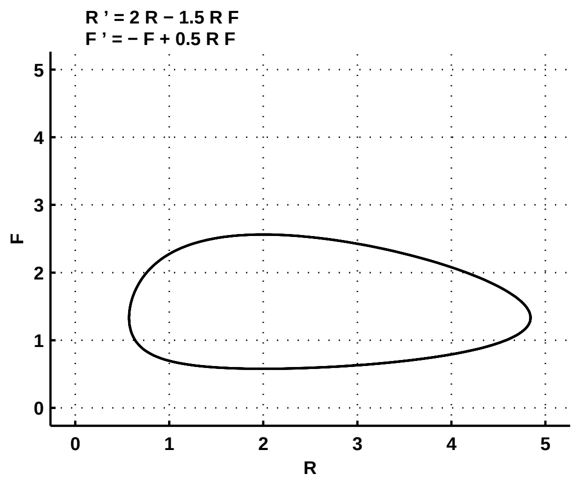
$$\frac{dx}{dt} = \left(a - \frac{1}{50}\right)x,$$
$$\frac{dy}{dt} = \frac{x}{50} + \left(a - \frac{1}{25}\right)y.$$

5. (10 points) Consider the following "predator-prey" model, where $R(t)$ denotes the population of rabbits and $F(t)$ denotes the population of foxes.

$$\frac{dR}{dt} = 2R - 1.5RF$$

$$\frac{dF}{dt} = -F + 0.5RF$$

A certain periodic solution to this system is plotted in the figure below.



The predator-prey model in Problem #5

Consider another solution, such that at some time t_0, the populations are given by $R(t_0) = 2$ and $F(t_0) = 2$. Will the population of rabbits ever exceed 5? Will the rabbits die out? Justify your answers.

Answer: Notice that (2, 2) is inside the closed solution curve plotted in the figure. By the uniqueness theorem for autonomous systems, the solution curve starting at (2, 2) cannot cross the plotted curve, and therefore is trapped inside. Since $R < 5$ everywhere on the plotted curve we know that the rabbit population will be less than 5 always.

Similarly $R > 1/2$ on the plotted curve so the rabbit population will satisfy the same inequality, and therefore will never die out.

6. (16 points) A credit card, issued by Duff beer company of Springfield to Homer Simpson charges 25% annually in interest on debt, compounded continuously. Suppose Homer has accumulated a debt of $6000 on his credit card. After getting a salary increase from Mr. Burns, Homer decides to repay his debt by making payments at the rate of $200 per month to the credit card company. He also decides not to make any more purchases.

a) Assuming that Homer's payments are made continuously, how long it will take him to repay his debt?

Answer: Homer's debt increases at the rate $0.25p = p/4$, and decreases at the annual rate of 2400. Hence the situation is modeled by the equation

$$P' = P/4 - 2400.$$

This linear equation can be solved by finding the integrating factor $e^{-t/4}$. Then

$$\left(e^{-t/4}P\right)' = e^{-t/4}P' - e^{-t/4}P/4 = -2400e^{-t/4}.$$

Integrating we get

$$e^{-t/4}P(t) = 9600e^{-t/4} + C \quad \text{or} \quad P(t) = 9600 + Ce^{t/4}.$$

The initial condition is

$$6000 = P(0) = 9600 + C.$$

Hence $C = -3600$. Our solution is

$$P(t) = 9600 - 3600e^{t/4}.$$

The debt will be repaid when $P(t) = 0$, or $e^{t/4} = 96/36 = 8/3$. Computing t we get $t = 4\ln(8/3) \approx 3.9233$ years.

b) Over the period that it takes to repay his debt, how much money will Homer have to pay to repay the debt together with the interest?

Answer: Homer will be paying $2,400 per year for $t = 4\ln(8/3) \approx 3.9233$ years. Thus he will have to pay

$$\$2,400 \times 4\ln(8/3) \approx \$9,416.$$

7. (16 points) Consider a spring with mass $m = 1$, and spring constant $k = 5$.

Answer: The equation for the spring is $my'' + \mu y' + ky = 0$. With the given parameters this is

$$y'' + \mu y' + 5y = 0.$$

The characteristic polynomial is $\lambda^2 + \mu\lambda + 5$. The roots are

$$\lambda = \frac{1}{2}\left[-\mu \pm \sqrt{\mu^2 - 20}\right].$$

a) Suppose that the damping constant is $\mu = 2$. Find the displacement of the spring as a function of time if the spring is started from spring-mass equilibrium with a velocity of 4.

Answer: With $\mu = 2$ the roots are

$$\lambda = -1 \pm \sqrt{-4} = -1 \pm 2i.$$

Therefore the general solution is

$$y(t) = e^{-t}\left(C_1 \cos(2t) + C_2 \sin(2t)\right).$$

Since the spring starts from equilibrium, we have $0 = y(0) = C_1$. The velocity is $y'(t) = C_2 e^{-t}(2\cos(2t) - \sin(2t))$. The initial velocity gives $4 = y'(0) = 2C_2$. Hence $C_2 = 2$, and the solution is

$$y(t) = 2e^{-t}\sin(2t).$$

b) What value of μ makes the spring critically damped?

Answer: The spring will be critically damped when the radicand $\mu^2 - 20 = 0$, or when

$$\mu = \sqrt{20}.$$