Math 211

Exam # 1

October 5, 1999

Part 2

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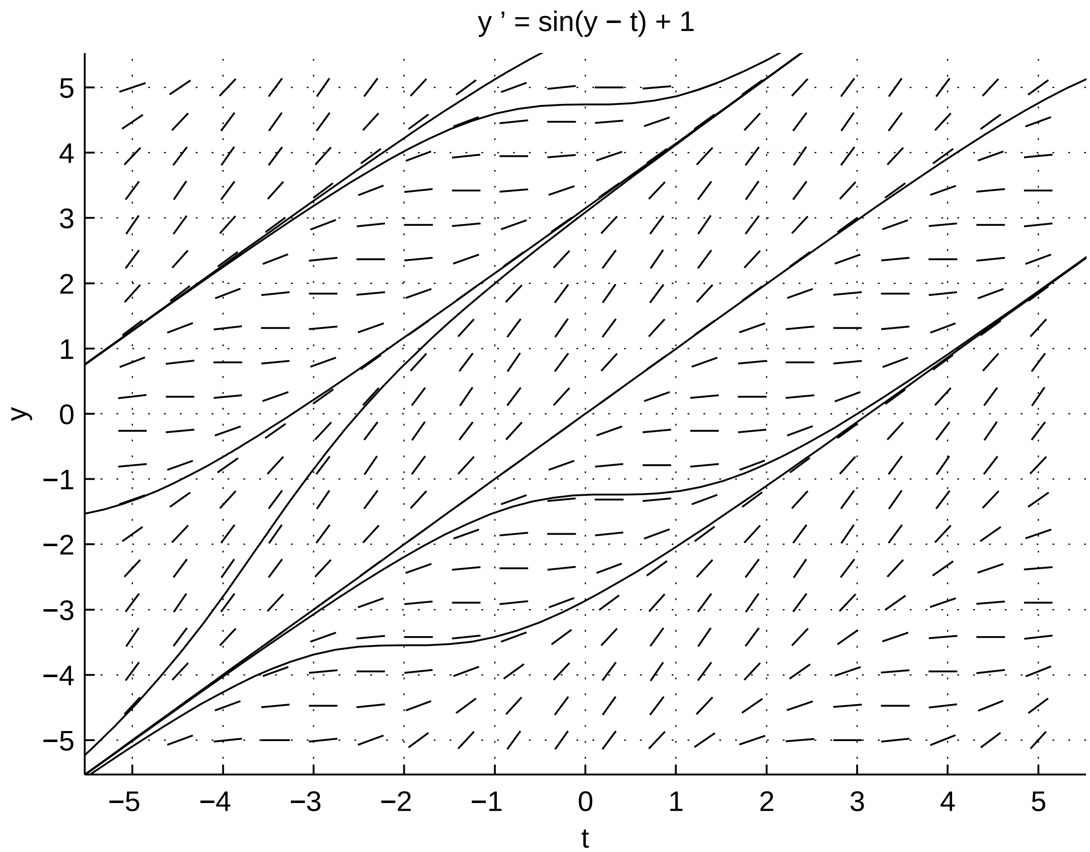
1. (12 points) Particular solutions to differential equations can sometimes be discovered by observing graphs of solutions. The simplest example is a *straight line solution*, which has a graph which is a straight line. `dfield5` is useful for making such discoveries. We will consider the example $y' = \sin(y - t) + 1$



Evidence of straight line solutions.

a) Enter the equation into `dfield5` and plot a few solutions to see if you can find evidence of straight line solutions. Submit a `dfield5` figure as your evidence, and state a conjecture as to the location and slopes of the straightline solutions.

Answer: The above figure shows evidence of straightline solutions, although none of the plotted solutions are actually straight lines. Some do come close, however. For example, there seems to be a straight line solution going through $(0, 0)$, and it has slope 1. There is clear visual evidence of other straight lines parallel to this one.

b) A solution whose graph is a straight line has the form $y = at + b$, where a and b are constants. Substitute $y = at + b$ into the differential equation and determine what a and b have to be in order that $y = at + b$ is a solution. Find all straight line solutions.

Answer: If $y(t) = at + b$, then $y' = a$. Plugging into the equation we get

$$a = \sin((at + b) - t) + 1 = \sin((a - 1)t + b) + 1.$$

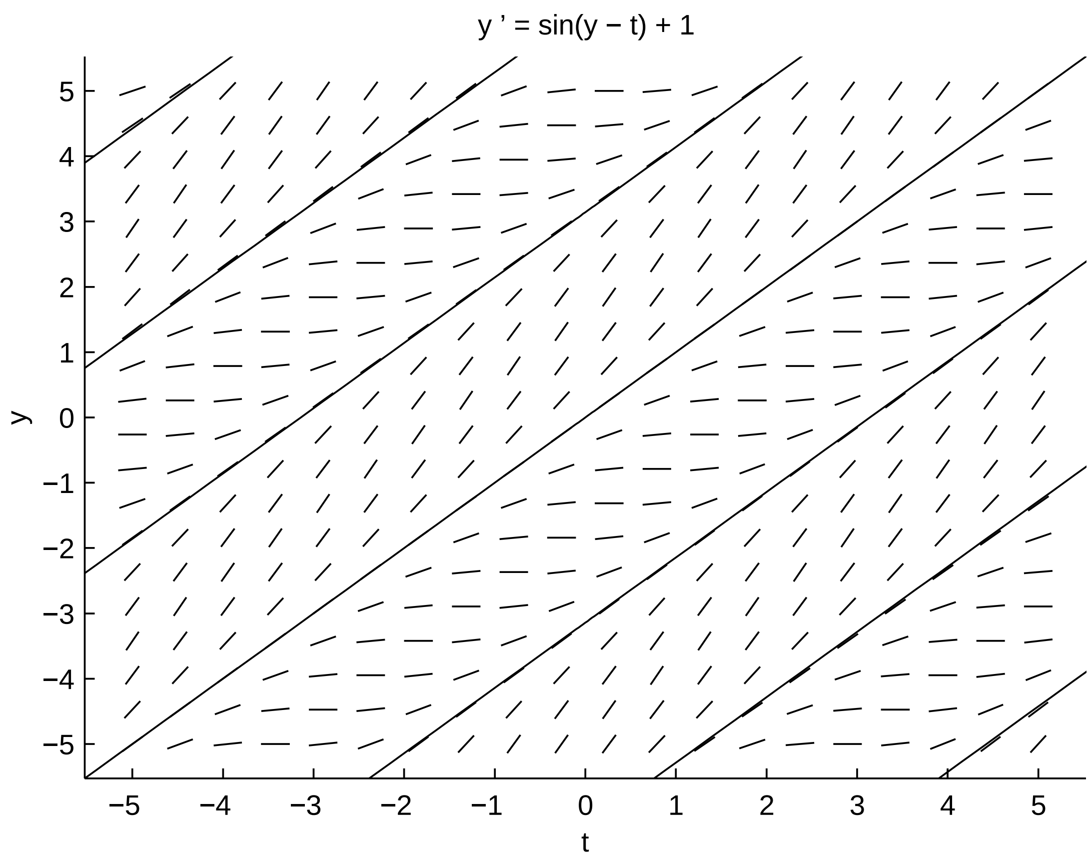
In order to have equality for all t the right hand side must be constant, which leads us to conclude that $a = 1$. The equation becomes

$$1 = \sin(b) + 1 \quad \text{or} \quad \sin(b) = 0.$$

This is satisfied if b is any integer multiple of π, i.e., $b = k\pi$ where k is any integer. The straight line solutions to our equation are of the form

$$y(t) = t + k\pi \quad k \text{ any integer.}$$

Thus there are infinitely many, all with slope 1.



Straight line solutions.

c) On a fresh `dfield5` Display Window (use Edit –> Erase all solutions) have `dfield5` draw some straightline solutions by entering the appropriate initial data.

Answer: Using keyboard input straight line solutions are found starting at $(0, k\pi)$ for a number of small integers k, as shown in the above figure.

2. (18 points) A population of fish in a lake is governed by the logistic equation

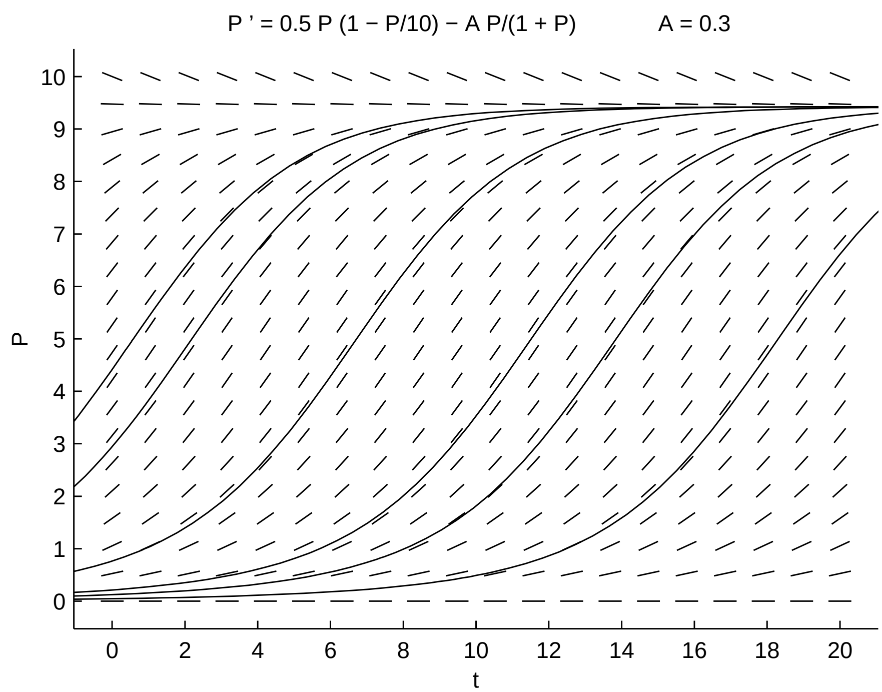
$$P' = 0.5P(1 - P/10).$$

Population is measured in units of 100,000 fish, and time is in years. The people living near the lake start fishing, and in the process remove $AP/(1+P)$ fish per year, where A is a constant. Hence the model becomes

$$P' = 0.5P(1 - P/10) - \frac{AP}{1+P}.$$

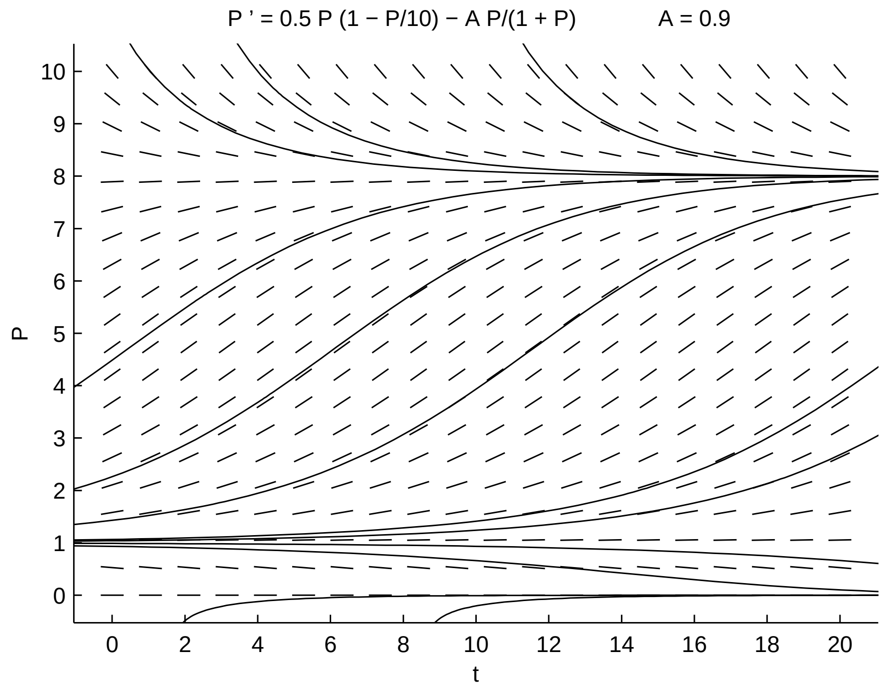
Your job is to discover all posible long term outcomes for the fish population for three values of A, $A = 0.3$, $A = 0.9$, and $A = 1.6$. For each of these cases do the following

a) Enter the equation into `dfield5` and plot enough solutions so that you can make a conjecture as to all possible outcomes. In the process you should conjecture as to the locations of the equilibrium points, and the type of each (i.e., are they stable or unstable?). Carefully state which initial conditions lead to which outcome. You should examine the populations over several time ranges to be sure that you do not miss interesting phenomena.



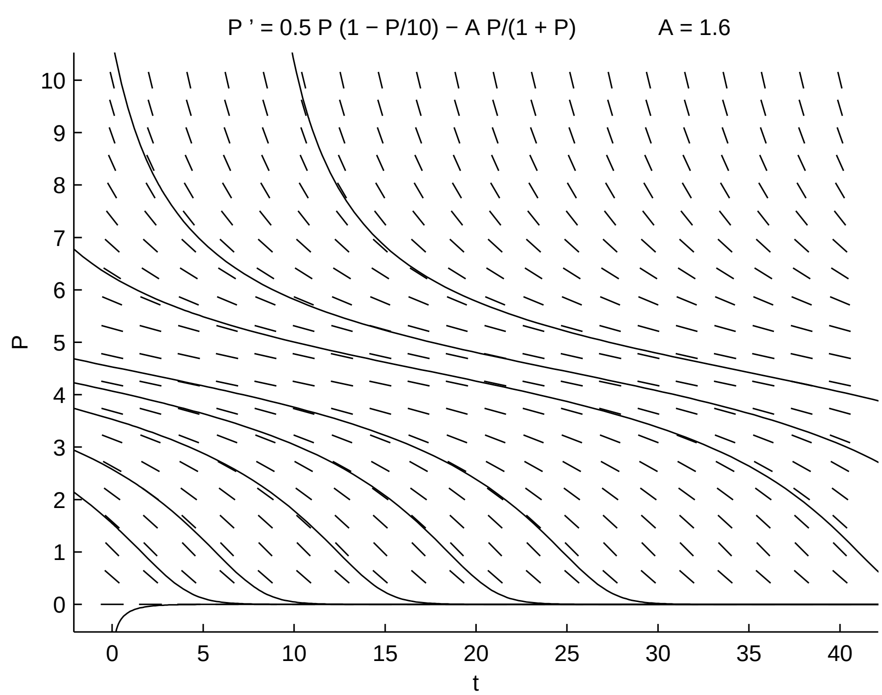
Population for $A = 0.3$.

Answer: For $A = 0.3$, the above figure shows several solution curves. These would seem to indicate that there is a stable equilibrium point at about $P = 9.4$, as well as the unstable one at $P = 0$.



Population for $A = 0.9$.

Answer: For $A = 0.9$, the above figure shows several solution curves. These would seem to indicate that there are two equilibrium points in addition to the one at $P = 0$. These seem to be close to $P = 1$ and $P = 8$. Now $P = 0$ seems to be a stable equilibrium point, $P = 1$ is unstable, and $P = 8$ is stable.



Population for $A = 1.6$.

Answer: For $A = 1.6$, the above figure shows several solution curves. These would seem to indicate that $P = 0$ is now the only equilibrium point, and it is stable.

b) Verify your conjecture by computing the equilibrium points *algebraically* and deciding if they are stable or unstable. We are only interested in realistic populations, so ignore possibilities that are negative or complex.

Answer: To find the equilibrium points we find where the right hand side of the equation equal to 0. This is

$$\begin{aligned} f(P) &= 0.5P(1 - P/10) - \frac{AP}{1+P} \\ &= \frac{P(10-P)(1+P) - 20AP}{20(1+P)} \\ &= \frac{P\left[(10-20A) + 9P - P^2\right]}{20(1+P)}. \end{aligned}$$

The equilibrium points are the roots of the numerator. These will be $P = 0$, and the roots of the quadratic $P^2 - 9P + (20A - 10) = 0$. We can now handle the three cases.

For $A = 0.3$ the quadratic is $P^2 - 9P - 4 = 0$. The roots are $(-9 \pm \sqrt{97})/2$. The roots compute to be $P = -0.399$ and $P = 9.399$. The only one of these that interests us is $P = 9.399$. There are a variety of ways to determine the types of the equilibrium points at $P = 0$ and $P = 9.399$. One way is to notice that $f(P)$ is negative for large values of P, and therefore for $P > 9.399$. Thus $f(P)$ goes from positive to negative as P increases through $P = 9.399$. This means that $f(P)$ is positive for $0 < P < 9.399$. Consequently $P = 0$ is unstable and $P = 9.399$ is stable.

For $A = 0.9$ the quadratic is $P^2 - 9P + 8 = 0$. Now the roots are $(9 \pm 7)/2$, or $P = 1$ and $P = 8$. Again $f(P)$ is negative for large values of P, and therefore for $P > 8$. $f(P)$ is positive for $2 < P < 8$ and negative for $0 < P < 2$. Thus $P = 0$ is stable, $P = 2$ is unstable and $P = 8$ is stable.

For $A = 1.6$ the quadratic is $P^2 - 9P + 22 = 0$. The roots are $(-9 \pm \sqrt{-7})/2$. Since these roots are complex we can ignore them, and the only equilibirium point is $P = 0$. $f(P)$ goes from positive to negative as P increases through 0, so $P = 0$ is stable.

3. (10 points) In this problem you are to examine possible outcomes of two species which are competing for resources. The model to be used is

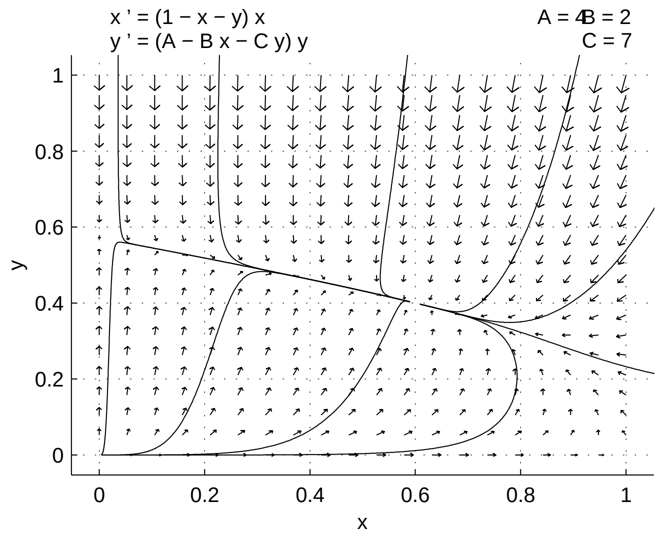
$$\begin{aligned} x' &= (1 - x - y)x \\ y' &= (A - Bx - Cy)y, \end{aligned}$$

where $x(t)$ and $y(t)$ are the populations of the two species. If you choose the menu item Gallery –> Competing Species in the `pplane5` Setup window, this system will be entered into `pplane5`. There are two cases to be considered

a) $A = 4$, $B = 2$, and $C = 7$. (This is the default.)

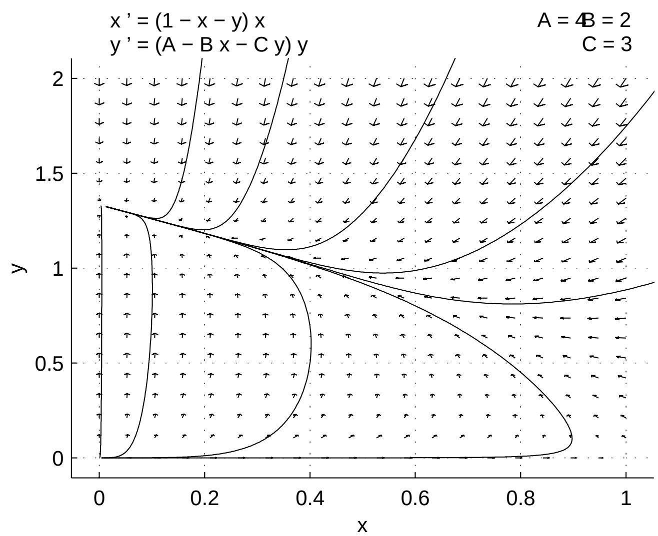
b) $A = 4$, $B = 2$, and $C = 3$.

For each of these cases plot enough solutions so that you can conjecture the long term behavior of all solutions. It may be necessary to change the size of the display window. For each case write a short paragraph describing your conjectures. Add `pplane5` plots as you feel necessary to provide evidence.



Competing species case a).

Answer: The above figure shows several solutions in the phase plane for case a). It seems as if all solutions in the positive quadrant approach the point $(0.6, 0.4)$ as $t \to \infty$.



Competing species case b).

Answer: the above figure shows several solutions in the phase plane for case b). It seems as if all solutions in the positive quadrant approach the point $(0, 1.33)$ (approximately) as $t \to \infty$. Thus in this case the population x dies out over time.