Math 211

Exam # 1

September 29, 1998

Part 2

Instructions: Part 2 of Exam #1 is an open book, open notes, untimed, take home exam. It is due in the Mathematics Department Office by 4:00 PM on Friday, October 2. You may not discuss the exam with your fellow students. If you have any questions, consult with one of the instructors for the course.

Write out and sign the pledge on your submitted solutions. Be sure to put your name on your submission. In addition put the name of your instructor at the top of the first page of your submission.

Please give reasons for all of your answers. Remember that some reasons are better than others. For example, it is better to refer to a theorem stated in class and/or in the books than to say "the computer printout shows that ...". Of course sometimes the computer printout is all you have.

In answering these questions you are not limited to the use of MATLAB, although it will be useful. You should use any combination of the analytic, qualitative, or numerical methods you have learned.

Your answers should consist of complete sentences organized into coherent paragraphs. This does not mean that they have to be long. Brevity is frequently a sign of understanding. MATLAB graphics should be numbered and referred to by that number. Any graphic which is not referred to will not be counted as part of your submission.

Notice that the points on the questions add to 40.

You will recognize that

$$p' = r\,p\left(1 - \frac{p}{K}\right).$$

is the logistic equation, and you will remember that it models a population when its resources are limited. The quantity r is the reproduction rate at small populations and K is the carrying capacity. Usually these parameters are constants, and for that case the analysis carried out class shows that for any solution $p(t)$ which has a positive initial value we have $p(t) \to K$ as $t \to \infty$. This is the reason that K is called the carrying capacity.

In some situations resources are not really limited. Consider, for example, the human population of the earth. In this case the intellectual development of the human race has allowed us to increase our capacity to grow food, which is probably the one resource which is most limiting to our continued growth. It is possible to improve upon the logistic model to allow for a change in the availability of resources over time.

One way to do this is to allow the carrying capacity to depend on time. For the carrying capacities in Problems 1, 2, & 3 you are to examine the long term behavior of solutions, especially in comparison to the carrying capacity. For each of these problems you are to:

a) Use `dfield5` to plot several solutions to the equation. (It is up to you to find a display window that is appropriate to the problem at hand.)

b) Based on the plot done in part a), describe the asymptotic behavior of the solutions to the equation. In particular, compare this asymptotic behavior to the asymptotic behavior of K.

Consider the equation in the following four cases:

1. (4 points) $K(t) = 1$, $r = 1$. This case is the standard logistic equation. Consequently, whatever the initial population, we expect that $p(t) \to K$ as $t \to \infty$. This case is here for comparison with the next two problems.

2. (9 points) $K(t) = 2 - e^{-t}$, $r = 1$. In this case $K(t)$ is monotone increasing, and $K(t)$ is asymptotic to 2 (i.e., $K(t) \to 2$ as $t \to \infty$). This might model a situation of a human population where, due to technological improvement, the availability of resources is increasing with time, although the increase is ultimately limited.

3. (9 points) $K(t) = 1 + t$, $r = 1$. Again $K(t)$ is monotone increasing, but this time it is unbounded, although of a very simple nature. In this case the solutions will be asymptotic to a function. You are not expected to find that function explicitly, but you should be able to describe it qualitatively.

4. (9 points) Consider a population of insects or small animals that is affected by the seasons. In such a case the available resources would vary periodically.

i) Using the ideas behind the models in Problems 2 & 3, give a model for this situation. To make things precise, assume that the unit of time is a year.

ii) Put your model into `dfield5` and plot a few solutions. Then describe the asymptotic behavior as you did in the previous problems.

5. (9 points) Consider a vibrating spring where the spring is “hard.” This means that the restoring force does not satisfy Hooke’s Law, but rather the restoring force has the form

$$F_s = -k(1 + \alpha y^2)y,$$

where y is the displacement. The constant k is still called the spring constant, and the constant $\alpha \geq 0$ measures the hardness of the spring. Notice that when $\alpha = 0$ Hooke’s law is satisfied. For the case when the mass is 1, $k = 3$, $\alpha = 0.5$, and there is no damping, put the equation for the hard vibrating spring into `pplane5`. Plot a few solutions, including the one where the spring starts with no velocity and a displacement of 2. Turn in a printout of the result.