7. Introduction to PPLANE6

A *planar system* is a system of differential equations of the form

$$\begin{aligned} x' &= f(t, x, y), \\ y' &= g(t, x, y). \end{aligned} \tag{7.1}$$

The variable t in system (7.1) usually represents time and is called the *independent* variable. Frequently, the right-hand sides of the differential equations do not explicitly involve the variable t, and can be written in the form

$$\begin{aligned} x' &= f(x, y), \\ y' &= g(x, y). \end{aligned} \tag{7.2}$$

Such a system is called *autonomous*.

The system

$$\begin{aligned} x' &= y, \\ y' &= -x, \end{aligned} \tag{7.3}$$

is an example of a planar autonomous system. The reader may check, by direct substitution, that the pair of functions $x(t) = \cos t$ and $y(t) = -\sin t$ satisfy both equations and is therefore a solution of system (7.3).

There are a number of ways that we can represent the solution of (7.3) graphically. The following code was used to construct the plots of x and y versus t in Figure 7.1.

```
>> t = linspace(0,4*pi);
>> x = cos(t); y = -sin(t);
>> subplot(2,1,1)
>> plot(t,x)
>> title('x = cos t'), ylabel('x')
>> subplot(2,1,2)
>> plot(t,y)
>> title('y = -sin t')
>> xlabel('t'), ylabel('y')
```

You can also plot the solution in the so-called *phase plane*. The commands

```
>> plot(x,y)
>> axis equal
>> title('The plot of y versus x')
>> xlabel('x'), ylabel('y)
```

produce a plot of y versus x in the phase plane, as shown in Figure 7.2.

System (7.1) can be written as the vector equation

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} f(t, [x, y]^T) \\ g(t, [x, y]^T) \end{bmatrix}. \tag{7.4}$$

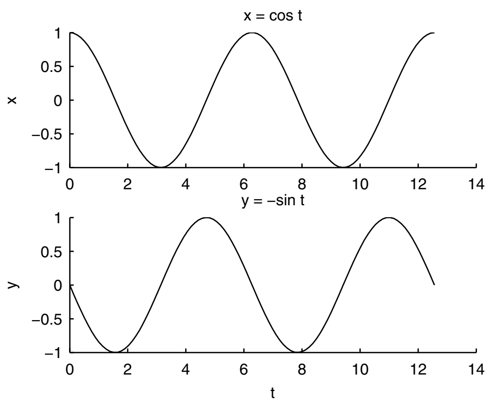

Figure 7.1. Plots of x and y versus t.

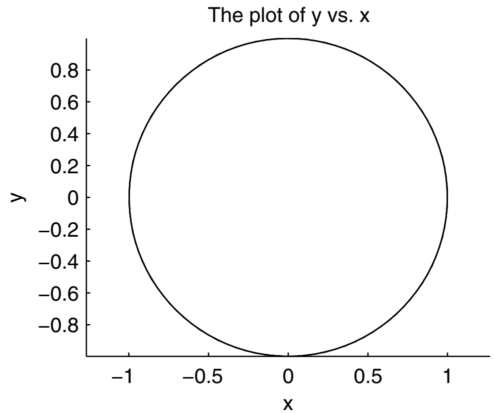

Figure 7.2. The solution in the phase plane.

If we let $\mathbf{x} = [x, y]^T$, then $\mathbf{x}' = [x', y']^T$ and equation (7.4) becomes

$$\mathbf{x}' = \begin{bmatrix} f(t, \mathbf{x}) \\ g(t, \mathbf{x}) \end{bmatrix}. \tag{7.5}$$

Finally, if we define $\mathbf{F}(t, \mathbf{x}) = [f(t, \mathbf{x}), g(t, \mathbf{x})]^T$, then equation (7.5) can be written as

$$\mathbf{x}' = \mathbf{F}(t, \mathbf{x}). \tag{7.6}$$

For a general planar system of the form $\mathbf{x}' = \mathbf{F}(t, \mathbf{x})$, a solution is a vector valued function $\mathbf{x}(t) = [x(t), y(t)]^T$. The individual components can be plotted as in Figure 7.1, or you can plot in the phase plane as shown in Figure 7.2. The fact that the function $\mathbf{x}(t)$ is a solution of the differential equation $\mathbf{x}' = \mathbf{F}(t, \mathbf{x})$ means that at every point $(t, \mathbf{x}(t))$, the curve $t \to \mathbf{x}(t)$ must have $\mathbf{F}(t, \mathbf{x}(t))$ as a tangent vector. For a fixed value of t we can imagine the vector $\mathbf{F}(t, \mathbf{x})$ attached to the point $\mathbf{x}$, representing the collection of all possible tangent vectors to solution curves for that specific value of t. Unfortunately this vector field changes as t changes, so this rather difficult visualization is not too useful. However, if the system is autonomous, then system (7.6) can be written

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}) \tag{7.7}$$

and the vector field $\mathbf{F}(\mathbf{x})$ does not change with time t. Therefore, for an autonomous system the same vector field represents all possible tangent vectors to solution curves for all values of t. If a solution curve is plotted parametrically, at each point the vector field must be tangent to the solution curve.

The MATLAB function `pplane6` makes this visualization easy.[1] This chapter provides an introduction to `pplane6`. We will delay discussing more advanced features of `pplane6` until you have learned more about systems of differential equations in the ensuing chapters. Actually, the functionality of `pplane6` is very similar to that of `dfield6`, so if you are familiar with `dfield6`, you will have no trouble with `pplane6`.

[1] To see if `pplane6` is installed properly on your computer enter `help pplane6`. If it is not installed see the Appendix to Chapter 3 for instructions on how to obtain it.

Starting PPLANE6

To see pplane in action, enter pplane6 at the MATLAB prompt. A new window will appear with the label PPLANE6 Setup. Figure 7.3 shows how this window looks on a PC running Windows. The appearance might differ slightly depending on your computer, but the functionality will be the same on all operating systems.

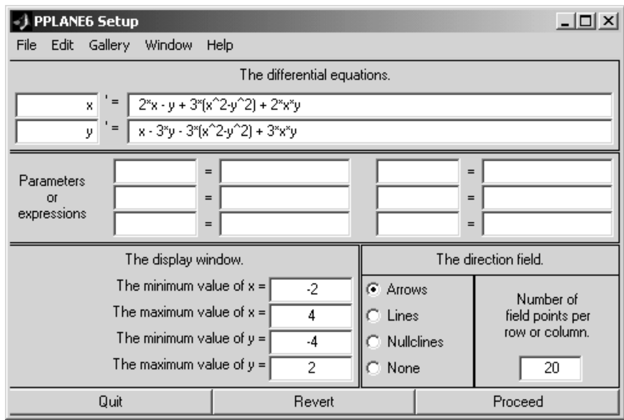

Figure 7.3. The setup window for pplane6.

You will notice that there is a rather complicated autonomous system already entered in the upper part of the PPLANE6 Setup window, in the form $x' = f(x, y)$ and $y' = g(x, y)$ of equation (7.2). There is a middle section for entering parameters and expressions, although none are entered at the moment. There is another section for describing a "display window," and yet another for defining what kind of field is to be used. There are are three buttons at the bottom of the window and several menus across the top. These are the same buttons and menus that are found in the DFIELD6 Setup window and they work in pplane6 just as they do in dfield6. We will describe the use of the menus in detail later, but for now leave everything unchanged and click the button labeled **Proceed**. After a few seconds another window will open, this one labeled PPLANE6 Display. An example of this window is shown in Figure 7.4.

The most prominent feature of the PPLANE6 Display window is a rectangle labeled with the variable x on the bottom, and the variable y on the left. The dimensions of this rectangle are slightly larger than the rectangle specified in the PPLANE6 Setup window in order to accommodate the extra space needed by the vectors in the field. Inside this rectangle the PPLANE6 Display window shows the vector field for the system defined in the PPLANE6 Setup window. At each point (x, y) of a grid of points, there is drawn a small arrow. The direction of the vector is the same as that of $\mathbf{F}(x, y) = [f(x, y), g(x, y)]^T$, entered as differential equations in the PPLANE6 Setup window, and the length varies with the magnitude of $\mathbf{F}(x, y)$. This vector must be tangent to any solution curve through (x, y). Simply said, the PPLANE6 Display window displays the phase plane for the planar system.

There are two buttons on the PPLANE6 Display window with the labels **Quit** and **Print**. There are several menus. Finally, below the vector field there is a small Cursor position window, and a larger message window through which pplane will communicate with the user. At this time it should contain the single word "Ready," indicating that it is ready to follow orders.

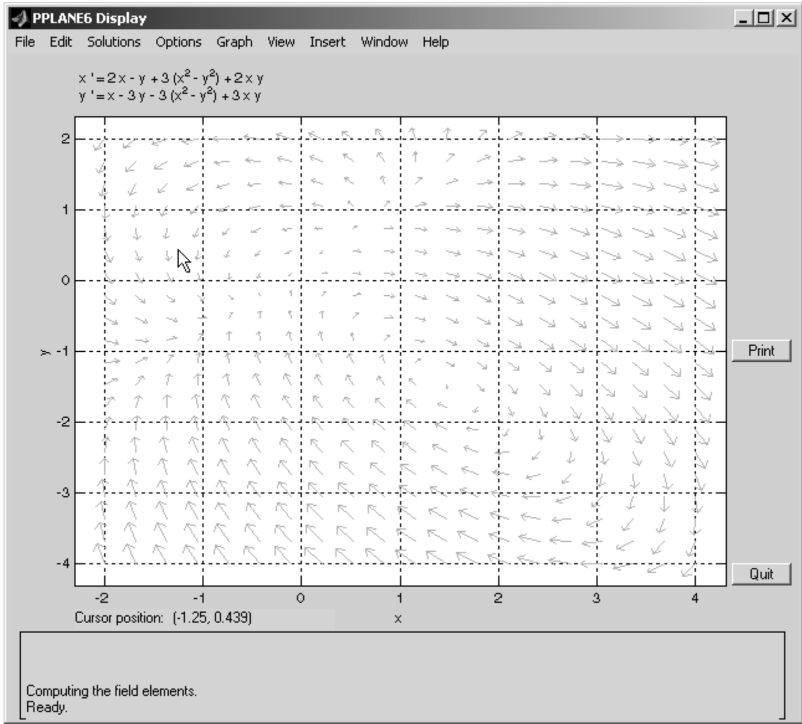

Figure 7.4. The display window for `pplane6`.

To compute and plot a solution curve from an initial point, move the mouse to that point, using the cursor position window to improve your accuracy, and click the mouse button. The solution will be computed and plotted, first in the direction in which the independent variable is increasing, and then in the opposite direction. After plotting several solutions, the display will look something like Figure 7.5. Notice that the solution curves all seem to start and stop in the same two points, and that the message window indicates that these are equilibrium points. We will have more to say about that later.

If you open the **Options** menu, you will see several ways of modifying how solutions are computed and plotted. The solution direction can be chosen by choosing **Options→Solution direction→Forward** or **Options→Solution direction→Back**. Some times it is desirable to indicate where the computation of a solution curve is started. This can be effected by selecting **Options→Mark initial points.**

In the **Edit** menu you will find the same "zoom in" and "zoom back" options available in `dfield6`.[2] Just as in `dfield6`, zooming can be done with the mouse.[3] For example, if you have a two button mouse, you can zoom at a point by clicking and and dragging with the right mouse button. There is an additional zoom option, **Zoom in square**, which we will describe in Chapter 13. You can print the PPLANE6 Display window to the default printer without the buttons and the message window by simply clicking the **Print** button. All of the print, save, and export options that are available for MATLAB figure windows are also available for the PPLANE6 Display window.

[2] See Chapter 3 for a review.
[3] See the inside cover of this manual for a summary of mouse button actions on various platforms.

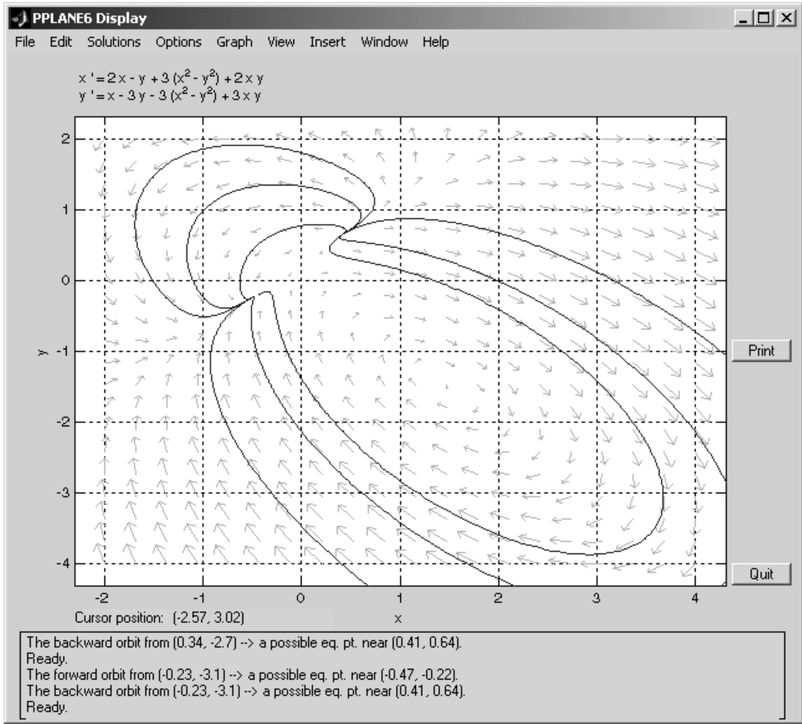

Figure 7.5. Several solutions to the equation.

Changing the System — Using the PPLANE6 Setup Window

We will illustrate the use of `pplane6` by using it to do a phase plane analysis of the motion of a pendulum. The differential equation for the motion of a pendulum is

$$mL\frac{d^2\theta}{dt^2} = -mg\sin(\theta) - c\frac{d\theta}{dt}, \tag{7.8}$$

where θ is the angular displacement of the pendulum from the vertical, L is the length of the pendulum arm, g is the acceleration due to gravity, and c is the damping constant. If we choose a convenient measure of time by setting $s = \sqrt{g/L}\,t$, the equation becomes[4]

$$\frac{d^2\theta}{ds^2} = -\sin(\theta) - a\frac{d\theta}{ds}, \tag{7.9}$$

where

$$a = \frac{c}{m\sqrt{gL}}$$

is again called the damping constant. Notice that this *scaling* of the variables reduces the number of parameters from four to one, and clearly indicates the importance of the *lumped* parameter a.

[4] Hint: $\frac{d\theta}{dt} = \frac{d\theta}{ds}\frac{ds}{dt} = \sqrt{\frac{g}{L}}\,\frac{d\theta}{ds}$; thus, $\frac{d^2\theta}{dt^2} = \frac{g}{L}\frac{d^2\theta}{ds^2}$ (why?).

We want to write this as a first order system, so we introduce the variables

$$x = \theta \quad \text{and} \quad y = \frac{d\theta}{ds}. \tag{7.10}$$

Then we have by (7.9) and (7.10)

$$\begin{aligned} \frac{dx}{ds} &= \frac{d\theta}{ds} = y, \\ \frac{dy}{ds} &= \frac{d^2\theta}{ds^2} = -\sin(\theta) - a\frac{d\theta}{ds} = -\sin(x) - ay, \end{aligned} \tag{7.11}$$

or, more simply,

$$\begin{aligned} x' &= y, \\ y' &= -\sin(x) - ay, \end{aligned} \tag{7.12}$$

where the prime indicates differentiation with respect to s. This is a planar autonomous system which we can analyze using `pplane6`. Enter the equations into the appropriate boxes the PPLANE6 Setup window This is exactly the same as it is in `dfield6`, except that there are two equations. See Figure 7.6.

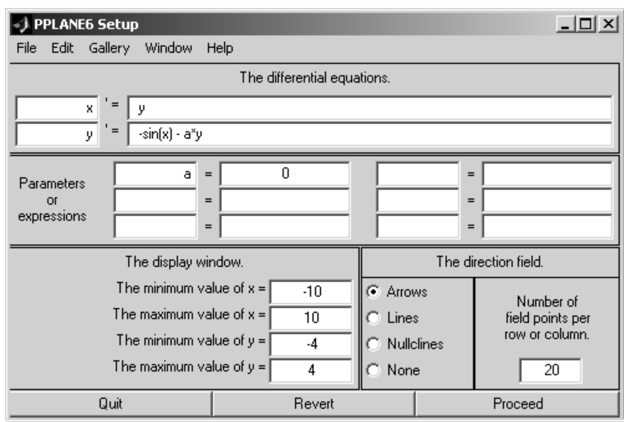

Figure 7.6. The PPLANE6 Setup window for the pendulum equation.

Since the damping constant a is not yet a number, we will assign it a value in one of the parameter boxes. For the time being, let's use the value $a = 0$. This value can be changed later to a positive number to see the phase plane for a damped pendulum. Next we have to describe the display rectangle. Since $x = \theta$ represents an angle, and we will want to plot a couple of full periods, we enter $-10 \le x \le 10$. For the dimensions in y we have to experiment. For now, $-4 \le y \le 4$ will do. Entering this data is just like it is in `dfield6`.

Finally, we decide which kind of direction field we want and how many field points we want displayed. These are available as menu options in `dfield6`. The choices are more important in `pplane6`, so they can be made directly in the PPLANE6 Setup window. Let's keep the default values. The completed PPLANE6 Setup window for the pendulum equation is shown in Figure 7.6. Click the **Proceed** button to transfer information from the PPLANE6 Setup window to the PPLANE6 Display window and start the computation of the direction field.

Plotting Solution Curves

Select **Options→Mark initial points** to mark the initial points of our solution curves. Select **Solutions→Keyboard input** and start a solution trajectory with initial condition $x(0) = 2$ and $y(0) = 0$. Note that this option behaves in a manner similar to that in `dfield6`. A closed orbit is plotted, giving the phase plane depiction of the standard motion of a pendulum shown in Figure 7.7. Go back to the PPLANE6 Setup window and change the parameter to $a = 0.5$, click proceed and then compute the solution as before. The result is shown in Figure 7.8.

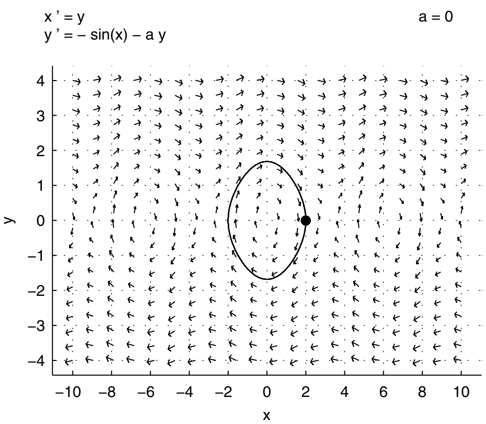

Figure 7.7. Phase-plane plot of the solution with initial condition $(x, y) = (2, 0)$ and $a = 0$.

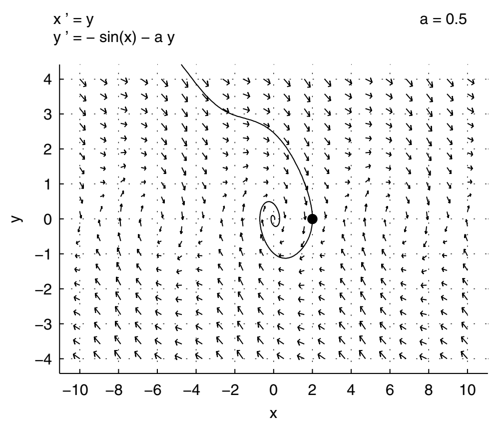

Figure 7.8. Phase-plane plot of the solution with initial condition $(x, y) = (2, 0)$ and $a = 0.5$.

Although the phase plane solutions pictured in Figures 7.7 and 7.8 are useful, we get a better understanding of a solution by looking at other graphical representations. This is easy to do in `pplane6`. Clicking on the **Graph** menu reveals five choices. Select **Graph→Both**. Notice that the vector field in the PPLANE6 Display window disappears and your cursor changes to a "cross hairs" as you move it over the figure. Center the cross hairs on the solution trajectory pictured in Figure 7.8 and click the (left) mouse button. This action should produce a plot in a new PPLANE6 t-plot window similar to that shown in Figure 7.9. Notice that both components of the solution are plotted against the independent variable t. In addition, this figure contains a legend identifying the two curves, and radio buttons that enable easy selection of five different views of the solution. These are the same options that appear in the **Graph** menu.

The plot in Figure 7.9 would be more interesting if the independent variable were restricted to lie between -2 and 10. We can accomplish this by cropping the figure. Position the cursor over the figure at $t = -2$. Then click the (left) mouse button, drag to $t = 10$, and release the mouse button. Notice that a green line appears to help you in this process. The **Crop** button is now enabled, and when you click it a new t-plot figure opens with the t variable limited to $-2 \le t \le 10$.

The most interesting of the plot options is the Composite plot. Click on that radio button, and you get Figure 7.10. This is basically a 3-dimensional plot of both components of the solution versus t. However, it also contains the other views as projections onto the three coordinate planes. The composite

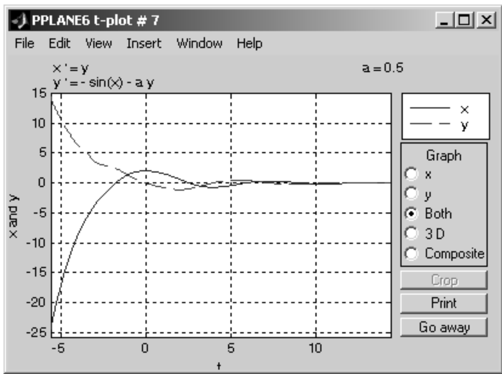

Figure 7.9. Plot of x and y versus t for the solution in Figure 7.8.

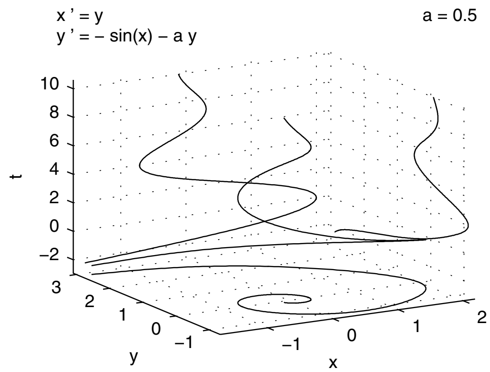

Figure 7.10. Composite plot of the solution.

plot lets you understand how the various graphical representations are related.

The composite and 3D views can be rotated so you can see them from different perspectives. Select **View→Figure Toolbar** to bring up the toolbar. Then select the rotate tool, the icon on the far right of the toolbar. Now click and drag in the figure window to rotate the view.

Other Properties of PPLANE6

Editing windows. The PPLANE6 Display window provides several editing commands that are similar to those in the DFIELD6 Display window. In fact there are a few more because of the additional features in `pplane6`. In particular, you can delete any graphics object, including text, by selecting **Edit→Delete a graphics object** and clicking on the object you wish to delete.

The PPLANE6 Display and t-plot figure windows can be edited just like any MATLAB figure window. However, use the property editors with caution, since they interfere with the interactive features of `pplane6`. The use of the rotate feature is an exception to this rule.

Printing, saving, and using the clipboard. You can print or export the `pplane6` windows in the standard ways described in Chapter2. The **Print** buttons on the PPLANE6 Display and t-plot windows will cause the figure to be printed to the default printer without the various buttons.

Saving and loading PPLANE6 systems and galleries. Again, as is the case for `dfield6`, the information on the setup window can be added to the **Gallery** menu, and can be saved in files. The files containing information about systems have the suffix `.pps`. You can also make, save, and load entirely new galleries that contain a number of important systems for a lecture or presentation. Gallery files have the suffix `.ppg`. The mechanisms for working with systems and galleries are the same as explained in Chapter 3 for `dfield6`.

Exercises

The planar, autonomous system having the form

$$x' = ax + by,$$
$$y' = cx + dy,$$

where a, b, c, and d are arbitrary real numbers, is called a *linear* system of first order differential equations. Exercises 1 – 6 each contain a solution of some linear system. Use MATLAB to create a plot of x versus t, y versus t, and a plot of y versus x in the phase plane. Use the `subplot` command, as in

```
subplot(221),plot(t,x),axis tight
subplot(222),plot(t,y),axis tight
subplot(212),plot(x,y),axis equal
```

to produce a plot containing all three plots. Use the suggested time interval.

1. $x = -2e^{-2t} + 3e^{-3t}$
   $y = 4e^{-2t} - 3e^{-3t}$
   $[-0.5, 2]$
2. $x = -4e^{2t} + 3e^{3t}$
   $y = -2e^{2t} + 3e^{3t}$
   $[-2, 0.5]$
3. $x = -2e^{-2t} - e^{3t}$
   $y = e^{-2t} + e^{3t}$
   $[-1.5, 1.0]$
4. $x = \cos 2t - 3\sin 2t$
   $y = 2\sin 2t + \cos 2t$
   $[-2\pi, 4\pi]$
5. $x = e^t(\cos t + 5\sin t)$
   $y = e^t(-8\sin t + \cos t)$
   $[-\pi, \pi/2]$
6. $x = e^{-t}(\cos t + \sin t)$
   $y = e^{-t}(\cos t - \sin t)$
   $[-\pi, \pi]$

For Exercises 7 – 12, select **Gallery→linear system** in the PPLANE6 Setup window. Adjust the parameters to match the indicated linear system. Accept all other default settings in the PPLANE6 Setup window. Select **Solutions→ Keyboard input** in the PPLANE6 Display window to start a solution trajectory with the given initial condition. Finally, select **Graph→Both** and click your solution trajectory to obtain plots of x and y versus t. For Exercise n, compare your output to Exercise $n - 6$ above.

7. $x' = -4x - y$
   $y' = 2x - y$
   $x(0) = 1, y(0) = 1$
8. $x' = x + 2y$
   $y' = -x + 4y$
   $x(0) = -1, y(0) = 1$
9. $x' = -7x - 10y$
   $y' = 5x + 8y$
   $x(0) = -3, y(0) = 2$
10. $x' = -2x - 4y$
    $y' = 2x + 2y$
    $x(0) = 1, y(0) = 1$
11. $x' = 4x + 2y$
    $y' = -5x - 2y$
    $x(0) = 1, y(0) = 1$
12. $x' = -x + y$
    $y' = -x - y$
    $x(0) = 1, y(0) = 1$
13. In the predator-prey system

$$L' = -L + LP,$$
$$P' = P - LP,$$

L represents a lady bug population and P represents a pest that lady bugs like to eat. Enter the system in the PPLANE6 Setup window, set the display window so that $0 \le L \le 2$ and $0 \le P \le 3$, then select **Arrows** for the direction field.

a) Use the Keyboard input window to start a solution trajectory with initial condition (0.5, 1.0). Note that the lady bug-pest population is periodic. As the lady bug population grows, their food supply dwindles and the lady bug population begins to decay. Of course, this gives the pest population time to flourish, and the resulting increase in the food supply means that the lady bug population begins to grow. Pretty soon, the populations come full cycle to their initial starting position.

b) Suppose that the pest population is harmful to a farmer's crop and he decides to use a poison spray to reduce the pest population. Of course, this will also kill the lady bugs. Question: Is this a wise idea? Adjust the system as follows:

$$L' = -L + LP - HL,$$
$$P' = P - LP - HP.$$

Note that this model assumes that the growth rate of each population is reduced by a fixed percentage of the population. Enter this system in the PPLANE6 Setup window, but keep the original display window settings. Create and set a parameter `H = 0.2`. Start a solution trajectory with initial condition $(0.5, 1.0)$.

c) Repeat part b) with $H = 0.4$, 0.6, and 0.8. Is this an effective way to control the pests? Why? Describe what happens to each population for each value of the parameter H.

14. Enter the system

$$x' = -\cos y + 2y \cos y^2 \cos 2x$$
$$y' = -\sin x + 2 \sin y^2 \sin 2x$$

in the PPLANE6 Setup window. Set the display window to $-10 \le x \le 10$ and $-2 \le y \le 4$ and select **None** for the vector field.

a) Select **File→Save the current system** and save the system with the name `teddybears.pps`.

b) Select **Gallery→linear system** to load the linear system template. Select **File→Load a system** and load the system `teddybears.pps`.

c) Select **Solutions→Keyboard input** and start a solution trajectory with initial condition $(\pi/2, 0)$ to create the "legs" of the Teddy bears, and a trajectory with initial condition $(-7.2, 3.5)$ to create the "heads" of the Teddy bears.

d) Experiment further with this "wild and wooly" example. Click the mouse in the phase plane to start other solution trajectories. Can you find the "eyes" of the Teddy bears?

When you portray sufficient solution trajectories in the phase plane so as to determine all of the important behavior of a planar, autonomous system, you have created what is called a "phase portrait." In Exercises 15 – 22, use `pplane6` to create a phase portrait for the indicated system on the prescribed display window. Take special notice of where solution curves end, as reported in the message window.

15. $x' = y$
$y' = (1 - y^2)y - x$
$-3 \le x \le 3,\ -3 \le y \le 3$

16. $x' = y + (y^2 - x^2 + 0.5x^4)(1 - x^2)$
$y' = x(1 - x^2) - y(y^2 - x^2 + 0.5x^4)$
$-3 \le x \le 3,\ -2 \le y \le 2$

17. $x' = x(2y^3 - x^3)$
$y' = -y(2x^3 - y^3)$
$-3 \le x \le 3,\ -3 \le y \le 3$

18. $x' = y + x^2 + xy$
$y' = -x + xy + y^2$
$-2 \le x \le 2,\ -2 \le y \le 2$

19. $x' = -y - x(x^2 - y^2)$
$y' = -x - y(x^2 - y^2))$
$-3 \le x \le 3,\ -3 \le y \le 3$

20. $x' = y - x(x^2 + y^2 - 1)$
$y' = -x - y(x^2 + y^2 - 1)$
$-2 \le x \le 2,\ -2 \le y \le 2$

21. $x' = x^2 - y^2$
$y' = 2xy$
$-1 \le x \le 1,\ - \le y \le 1$

22. $x' = x^3 - 3xy^2$
$y' = 3xy^2 - y^3$
$-1 \le x \le 1,\ -1 \le y \le 1$

The equation $my'' + dy' + ky = 0$ represents a damped, spring-mass system. In Exercises 23 – 26, values of the parameters m, d, and k have been chosen to create a specific example of a damped, spring-mass system. We are looking for the solution y with the given initial position and velocity. Let

$$x_1 = y, \quad \text{and} \quad x_2 = y',$$

and use this change of variables to write each spring-mass system as a planar, autonomous system. Use `pplane6` to obtain a printout of the graph of y versus t.

23. $y'' + 2y' + 2y = 0$
$y(0) = 0,\ y'(0) = -1$

24. $y'' + 4y' + 4y = 0$
$y(0) = -2,\ y'(0) = 2$

25. $y'' + 3y' + 2y = 0$
$y(0) = 3,\ y'(0) = 2$

26. $y'' + 4y = 0$
$y(0) = -4,\ y'(0) = -2$

27. Tanks A and B contain mixtures of water and a pollutant. Tank A has a 100 gallon capacity and is full. Tank B is also full and has a 200 gallon capacity. Initially, tank A contains 40 pounds of pollutant and tank B contains pure water (no pollutant). At $t = 0$, pure water begins pouring into tank A at a rate of 10 gallons per minute. The mixture flows from tank A into tank B at a rate of 10 gallons per minute, and drains out of tank B at 10 gallons per minute. If x_A and x_B represent the number of pounds of pollutant in tank A and tank B, respectively, show that

$$x_A' = -\frac{1}{10}x_A,$$
$$x_B' = \frac{1}{10}x_A - \frac{1}{20}x_B,$$

where $x_A(0) = 40$ and $x_B(0) = 0$.

a) Use `pplane6` and its zoom tools to estimate the maximum amount of pollutant in tank B and the time that it occurs.

b) Use `pplane6` to estimate the time that it takes the pollutant in tank B to reach a level of 10 pounds. *Note:* This event occurs at two separate times, once as the pollutant level is rising in tank B, and a second time when the pollutant level is decreasing.

28. Fasten a length of string to a hook. Securely fasten a mass to the the other end of the string. Let θ represent the displacement of the mass from the vertical. Assume that θ is measured in radians, with positive displacements in the counterclockwise direction. Let ω represent the angular velocity of the mass, with positive angular velocity in the counterclockwise direction. Displace the mass 30° counter-clockwise (positive $\pi/6$ radians) from the vertical and release the mass from rest. Let the pendulum decay to a stable equilibrium point at $\theta = 0$, $\omega = 0$.

Important Note. You will not benefit as much as you should from this exercise if you don't first complete parts a) and b) before attempting part c).

a) Without using any technology, sketch graphs of θ versus t and ω versus t approximating the motion of the pendulum.

b) Without using any technology, sketch graphs of ω versus θ. Place ω on the vertical axis, θ on the horizontal axis. *Note:* This is a lot harder than it looks. We suggest that you work with a partner or a group and compare solutions before moving on to part c).

c) Select **Gallery→pendulum** in the PPLANE6 Setup window. Adjust the damping parameter to `D=0.1`, and set the display window so that $-1 \le \theta \le 1$ and $-1 \le \omega \le 1$. Select **Options→Solution direction→Forward** and use the Keyboard input window to start a solution trajectory with initial condition $\theta(0) = \pi/6$ and $\omega(0) = 0$. Compare this result with your hand-drawn solution in part b). Select **Graph→Both** and click your solution trajectory in the phase plane to produce plots of θ versus t and ω versus t. Compare these with your hand-drawn solutions in part a).

29. Repeat the pendulum experiment of the previous problem. Displace the mass 30° counter-clockwise (positive $\pi/6$ radians) from the vertical, only this time do not release the mass from rest. Instead, push the mass in the clockwise (negative) direction with enough negative angular velocity so that it spins around in a circle exactly one time before settling into a motion decaying to a stable equilibrium. The tricky part of this experiment is the fact that the stable equilibrium point is now $\theta = -2\pi$, $\omega = 0$.

a) Without using any technology, sketch graphs of θ versus t and ω versus t approximating the motion of the pendulum.

b) Without using any technology, sketch graphs of ω versus θ. Place ω on the vertical axis, θ on the horizontal axis. *Note:* This is a lot harder than it looks. We suggest that you work with a partner or a group and compare solutions before moving on to part c).

c) Select **Gallery→pendulum** in the PPLANE6 Setup window. Adjust the damping parameter to `D=0.1`, and set the display window so that $-10 \le \theta \le 5$ and $-4 \le \omega \le 4$. Select **Options→Solution**

direction→Forward and use the Keyboard input window to start a solution trajectory with initial condition $\theta(0) = \pi/6$ rad and $\omega(0) = -2.5$ rad/s. Compare this result with your hand-drawn solution in part b). Select **Graph→Both** and click your solution trajectory in the phase plane to produce plots of θ versus t and ω versus t. Compare these with your hand-drawn solutions in part a).

30. Consider the predator-prey system

$$R' = R - RF,$$
$$F' = -F + RF,$$

where R and F represent rabbit and fox populations, respectively. Enter the system in `pplane6` and set the display window so that $0 \le R \le 2$ and $0 \le F \le 2$. Use Keyboard input to start a solution trajectory at $R = 0.5$ and $F = 1$. Note that the trajectory is periodic. Use `pplane6` to estimate the time it takes to travel this periodic trajectory exactly once; i.e., find the period of the oscillation. *Hint:* Try cropping an `x` versus `t` plot.

31. Start a new session by exiting and restarting `pplane6`. Select **File→Delete the current gallery** in the PPLANE6 Setup window. Examine your textbook and select three planar, autonomous systems of interest. Enter the first system and set the parameters (if any), then adjust the display window and select the direction field. Select **Gallery→Add current system to the gallery** to add your system to the gallery. Add each of your remaining systems to the gallery, then select **File→Save a gallery** to create and save your new gallery.

    a) Start a new session by exiting and restarting `pplane6`. Delete the current gallery, then load your newly created gallery.

    b) Check that each system in the gallery works as it should.