3. Introduction to DFIELD6

A first order ordinary differential equation has the form

$$x' = f(t, x).$$

To solve this equation we must find a function $x(t)$ such that

$$x'(t) = f(t, x(t)), \qquad \text{for all } t.$$

This means that at every point $(t, x(t))$ on the graph of x, the graph must have slope equal to $f(t, x(t))$.

We can turn this interpretation around to give a geometric view of what a differential equation is, and what it means to solve the equation. At each point (t, x), the number $f(t, x)$ represents the slope of a solution curve through this point. Imagine, if you can, a small line segment attached to each point (t, x) with slope $f(t, x)$. This collection of lines is called a *direction line field*, and it provides the geometric interpretation of a differential equation. To find a solution we must find a curve in the plane which is tangent at each point to the direction line at that point.

Admittedly, it is difficult to visualize such a direction field. This is where the MATLAB routine `dfield6` demonstrates its value.[1] Given a differential equation, it will plot the direction lines at a large number of points — enough so that the entire direction line field can be visualized by mentally interpolating between the field elements. This enables the user to get some geometric insight into the solutions of the equation.

Starting DFIELD6

To see `dfield6` in action, enter `dfield6` at the MATLAB prompt. After a short wait, a new window will appear with the label DFIELD6 Setup. Figure 3.1 shows how this window looks on a PC running Windows. The appearance will differ slightly depending on your computer, but the functionality will be the same on all machines.

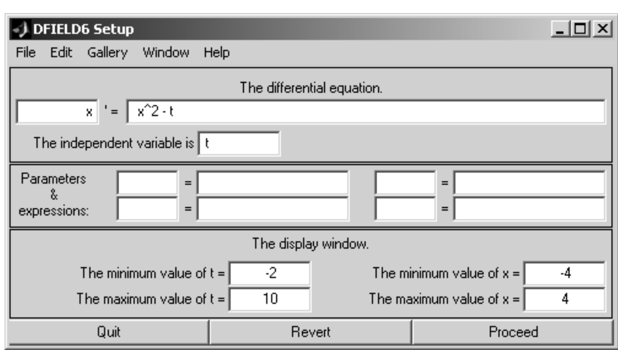

Figure 3.1. The setup window for `dfield6`.

[1] The MATLAB function `dfield6` is not distributed with MATLAB. To discover if it is installed properly on your computer enter `help dfield6` at the MATLAB prompt. If it is not installed, see the Appendix to this chapter for instructions on how to obtain it.

The DFIELD6 Setup window is an example of a MATLAB *figure window*. We have already seen figure windows in Chapter 2, but this one looks very different, so we see that a figure window can assume a variety of forms. In a MATLAB session there will always be one command window open on your screen and perhaps a number of figure windows as well.

The equation $x' = x^2 - t$ is entered in the edit box entitled “The differential equation” of the DFIELD6 Setup window. There is also an edit box for the independent variable and several edit boxes are available for parameters. The default values in “The display window” limit the independent variable t to $-2 \leq t \leq 10$ and the dependent variable x to $-4 \leq x \leq 4$. At the bottom of the DFIELD6 Setup window there are three buttons labelled **Quit**, **Revert**, and **Proceed**.

The Direction Field

We will describe the setup window in detail later, but for now click the button with the label **Proceed**. After a few seconds another window will appear, this one labeled DFIELD6 Display. An example of this window is shown in Figure 3.2.

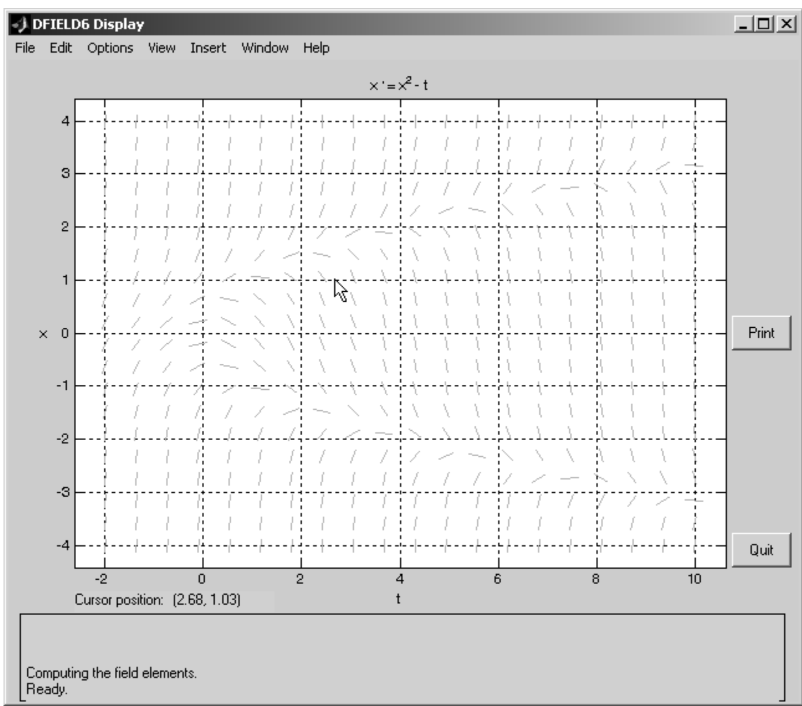

Figure 3.2. The display window for `dfield6`.

The most prominent feature of the DFIELD6 Display window is a rectangular grid labeled with the differential equation $x' = x^2 - t$ on the top, the independent variable t on the bottom, and the dependent variable x on the left. The dimensions of this rectangle are slightly larger than the rectangle specified in the DFIELD6 Setup window to accommodate the extra space needed by the direction field lines. Inside this rectangle is a grid of points, 20 in each direction, for a total of 400 points. At each such point with coordinates (t, x) there is shown a small line segment centered at (t, x) with slope equal to $x^2 - t$. This collection of lines is a subset of the direction field.

There is a pair of buttons on the DFIELD6 Display window: **Quit** and **Print**. There are several menus: File, Edit, Options, Insert, and Help. Below the direction field there is a small window giving the coordinates of the cursor, and a larger message window through which `dfield6` will communicate with us. Note that the last line of this window contains the word "Ready," indicating that `dfield6` is ready to follow orders.

Initial Value Problems

The differential equation $x' = x^2 - t$ is in *normal form,* meaning that the derivative x' is expressed as a function of the independent variable t and the dependent variable x. You will notice from Figure 3.1 that `dfield6` requires the differential equation to be in normal form. Most differential equations have infinitely many solutions. In order to get a particular solution it is necessary to specify an initial condition. The differential equation with initial condition,

$$x' = f(t, x), \quad x(t_0) = x_0, \tag{3.1}$$

is called an *initial value problem.*

A *solution curve* of a differential equation $x' = f(t, x)$ is the graph of a function $x(t)$ which solves the differential equation. In particular we get a solution curve by computing and plotting the solution to an initial value problem. This is an easy process using `dfield6`. With the differential equation in normal form, we enter it and the other data in the setup window (see Figure 3.1). We then proceed to the display window (Figure 3.2). To solve with a given initial condition $x(t_0) = x_0$, we move the mouse to the point (t_0, x_0), using the cursor position display at the bottom of the figure to improve our accuracy, and then click the mouse button. The computer will compute and plot the solution through the selected point, first in the direction in which the independent variable is increasing (the "Forward" direction), and then in the opposite direction (the "Backward" direction). The result should be something like Figure 3.3. After computing and plotting several solutions, the display might look something like that shown in Figure 3.4.

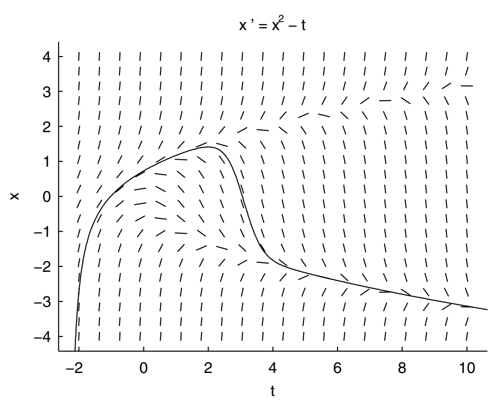

Figure 3.3. A solution of the ODE $x' = x^2 - t$.

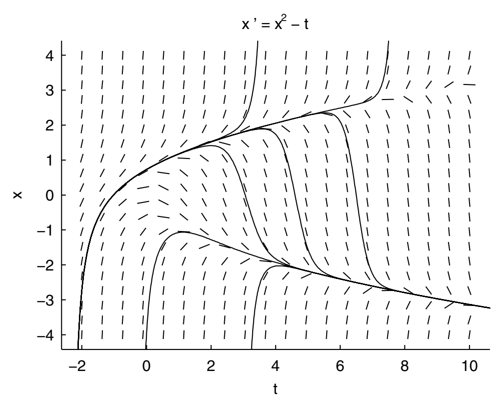

Figure 3.4. Several solutions of the ODE $x' = x^2 - t$.

Finer Control of Data. The next example illustrates several features of `dfield6`, including how to be accurate with initial conditions.

Example 1. *The voltage y on the capacitor in a certain RC circuit is modeled by the differential equation $y' + y = 3 + \cos x$, where we are using the variable x to represent time. Use* `dfield6` *to plot the voltage over the interval $0 \le x \le 20$, assuming that $y(0) = 1$.*

You will notice that we are asked to solve the initial value problem

$$y' + y = 3 + \cos x, \quad y(0) = 1. \tag{3.2}$$

The dependent variable in this example is y and the independent variable is x. The differential equation $y' + y = 3 + \cos x$ is not in normal form, so we put it in normal form by solving the equation for y', getting $y' = -y + 3 + \cos x$.

Return to the DFIELD6 Setup window and select **Edit→Clear all**.[2] Notice that there are options on the Edit menu to clear particular regions of the DFIELD6 Setup window and each of these options possesses a keyboard accelerator. Enter the left and right sides of the differential equation $y' = -y + 3 + \cos x$, the independent variable (x in this case), and define the display window in the DFIELD6 Setup window as shown in Figure 3.5.[3]

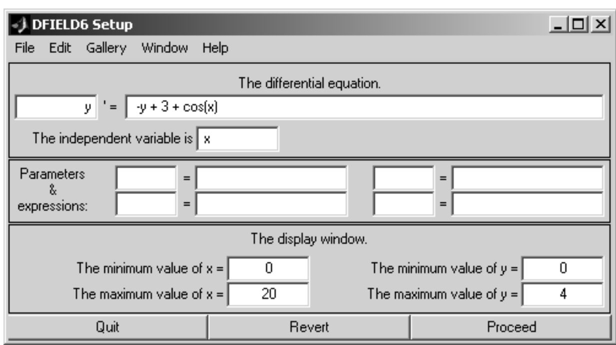

Figure 3.5. The setup window for $y' = -y + 3 + \cos x$.

Should your data entry become hopelessly mangled, click the **Revert** button to restore the original entries. The initial value problem in (3.2) contains no parameters, so leave the parameter fields in the DFIELD6 Setup window blank. Click the **Proceed** button to transfer the information in the DFIELD6 Setup window to the DFIELD6 Display window and start the computation of the direction field.

Choosing the initial point for the solution curve with the mouse is convenient, but it is often difficult to be accurate, even with the help of the cursor position display. Instead, in the DFIELD6 Display window, select **Options→Keyboard input**. Enter the initial condition, $y(0) = 1$, as shown in Figure 3.6. Click the **Compute** button in the DFIELD6 Keyboard input window to compute the trajectory shown in Figure 3.7.

[2] We continue to use the notation **Edit→Clear all** to signify that you should select "Clear all" from the Edit menu.

[3] MATLAB is case-sensitive. Thus, the variable `Y` is completely different from the variable `y`.

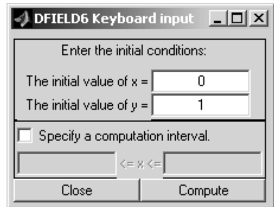

Figure 3.6. The initial condition $y(0) = 1$ starts the solution trajectory at (0, 1).

Notice that it is not necessary to specify a computation interval. However, you can specify one if you wish by clicking the “Specify a computation interval” checkbox in the DFIELD6 Keyboard Input window (See Figure 3.6), and then filling in the starting and ending times of the desired solution interval. For example, start a solution trajectory with initial condition $y(0) = 0$, but set the computation interval so that $-\pi \leq x \leq \pi$. Try it!

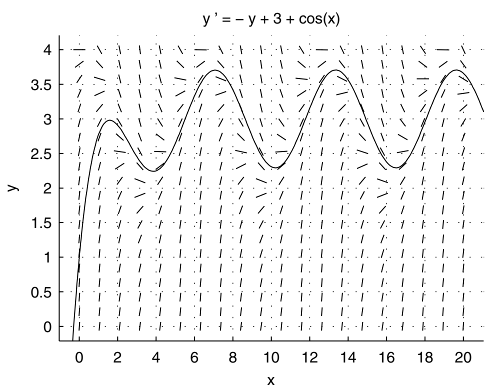

Figure 3.7. Solution of $y' + y = 3 + \cos x$, $y(0) = 1$.

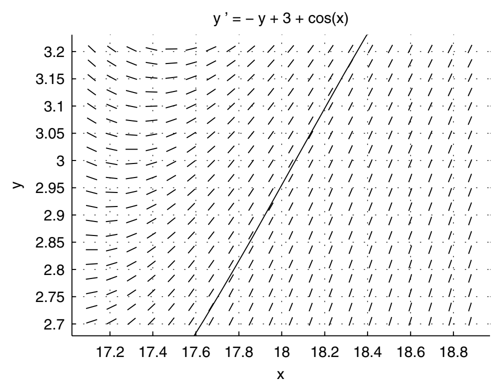

Figure 3.8. Zooming in to find $y(18)$.

Example 2. *For the voltage $y(x)$ computed in Example 1 find $y(18)$, accurate to 2 decimal places.*

From the graph of the solution in Figure 3.7 we can see that the voltage $y(18)$ is approximately 3. However, this is not accurate enough. To get more accuracy, we will increase the resolution using the zoom tools in `dfield6`. Select **Edit→Zoom in** in the DFIELD6 Display window, then single-click the (left) mouse button in the DFIELD6 Display window near the point (18, 3). Additional “zooms” require that you revisit **Edit→Zoom in** before clicking the mouse button to zoom. There is a faster way to zoom in that is platform dependent. If you have a mouse with more than one button, click the right mouse button at the zoom point (or control-click the left mouse button at the zoom point). On a Macintosh, option-click the mouse button at the zoom point.[4] After performing a couple of zooms (results may vary

[4] Mouse actions are platform dependent in `dfield6`. See the front and back covers of this manual for a summary of mouse actions on various platforms.

on your machine), greater resolution is obtained. When you reach a point similar to Figure 3.8, you can use the cursor position display to see that $y(18) \approx 2.96$.

Existence and Uniqueness

It would be comforting to know in advance that a solution of an initial value problem exists, especially if you are about to invest a lot of time and energy in an attempt to find a solution. A second (but no less important) question is uniqueness: is there only one solution? Or does the initial value problem have more than one solution? Fortunately, existence and uniqueness of solutions have been thoroughly examined and there is a beautiful theorem that we can use to answer these questions.

Theorem 1. *Suppose that the function $f(t, x)$ is defined in the rectangle R defined by $a \le t \le b$ and $c \le x \le d$. Suppose also that f and $\partial f/\partial x$ are both continuous in R. Then, given any point $(t_0, x_0) \in R$, there is one and only one function $x(t)$ defined for t in an interval containing t_0 such that $x(t_0) = x_0$ and $x' = f(t, x)$. Furthermore, the function $x(t)$ is defined both for $t > t_0$ and for $t < t_0$, at least until the graph of x leaves the rectangle R through one of its four edges.*[5]

Example 3. *Use* `dfield6` *to sketch the solution of the initial value problem*

$$x' = x^2, \quad x(0) = 1.$$

Set the display window so that $-2 \le t \le 3$ and $-4 \le x \le 4$.

Enter the differential equation $x' = x^2$, the independent variable t, and the display window ranges $-2 \le t \le 3$ and $-4 \le x \le 4$ in the DFIELD6 Setup window. Click **Proceed** to compute the direction field. Select **Options→Keyboard input** in the DFIELD6 Display window and enter the initial condition $x(0) = 1$ in the DFIELD6 Keyboard input window. If all goes well, you should produce an image similar to that in Figure 3.9.

The differential equation $x' = x^2$ is in the form $x' = f(t, x)$, with $f(t, x) = x^2$. In addition, $f(t, x) = x^2$ and $\partial f/\partial x = 2x$ are continuous on the rectangle R defined by $-2 \le t \le 3$ and $-4 \le x \le 4$. Therefore, Theorem 1 states that the initial value problem has a solution as shown in Figure 3.9, and that this solution is unique. Use the mouse to experiment. You will see that any other solution is parallel to the first one and does not pass through the point (0, 1).

Theorem 1 does not make a definitive statement about the domain of a solution. For example, does the solution in Figure 3.9 exist for all t or does it reach positive infinity in a finite amount of time? This question cannot be answered by `dfield6` alone, although it can provide a hint. Go back to the Setup window and change the display window to $0 \le t \le 1.5$ and $0 \le x \le 40$ in order to focus on the solution in Figure 3.9 near $t = 1$. When we proceed to the display window and recompute the solution, we get the result shown in Figure 3.10. This seems to indicate that the solution becomes infinite near $t = 1$. To check this out, we solve the differential equation. The general solution is $x(t) = 1/(C - t)$, where C is an arbitrary constant. Substituting the initial condition $x(0) = 1$ into the equation $x(t) = 1/(C - t)$,

[5] The notation $\partial f/\partial x$ represents the *partial derivative of f with respect to x*. Suppose, for example, that $f(t, x) = x^2 - t$. To find $\partial f/\partial x$, think of t as a constant and differentiate with respect to x to obtain $\partial f/\partial x = 2x$. Similarly, to find $\partial f/\partial t$, think of x as a constant and differentiate with respect to t to obtain $\partial f/\partial t = -1$.

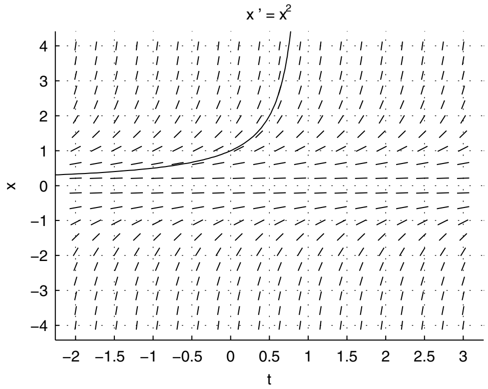

Figure 3.9. The solution of $dx/dt = x^2$, $x(0) = 1$ is unique.

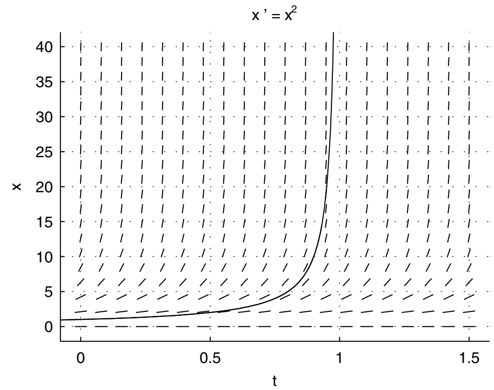

Figure 3.10. The solution "blows up" at $t = 1$.

we find that $1 = 1/(C - 0)$, or $C = 1$, so the solution is $x(t) = 1/(1 - t)$. From this equation, we see that $\lim_{t \to 1^-} x(t) = +\infty$. Mathematicians like to say that the solution "blows up" at $t = 1$.[6] In this particular case, if the independent variable t represents time (in seconds), then the solution trajectory reaches positive infinity when one second of time elapses.

Example 4. *Consider the differential equation*

$$\frac{dx}{dt} = x^2 - t.$$

Sketch solutions with initial conditions $x(2) = 0$, $x(3) = 0$, *and* $x(4) = 0$. *Determine whether or not these solution curves intersect in the display window defined by* $-2 \le t \le 10$ *and* $-4 \le x \le 4$.

Go to the setup window and select **Gallery→default equation**. The correct data will be entered. Click **Proceed** to transfer this information and begin computation of the direction field in the DFIELD6 Display window. Select **Options→Keyboard input** in the DFIELD6 Display window and compute solutions for each of the initial conditions $x(2) = 0$, $x(3) = 0$, and $x(4) = 0$. If all goes well, you should produce an image similar to that in Figure 3.11.

The ODE $x' = x^2 - t$ is in normal form, $x' = f(t, x)$, with $f(t, x) = x^2 - t$. Both f and $\partial f/\partial x = 2x$ are continuous on the display window defined by $-2 \le t \le 10$ and $-4 \le x \le 4$. In Figure 3.11, it appears that the solution trajectories merge into one trajectory near the point (6, −2.4) (or perhaps even sooner). However, Theorem 1 guarantees that solutions cannot cross or meet in the display window of Figure 3.11.

This situation can be analyzed by zooming in near the point (6, −2.4). After performing numerous zooms, some separation in the trajectories begins to occur, as shown in Figure 3.12. Without Theorem 1, we might have mistakenly assumed that the trajectories merged into one trajectory.

It is also possible to zoom in by dragging a "zoom box". If you have a two button mouse, this can be done by depressing the right mouse button, then dragging the mouse. Once the zoom box is drawn

[6] The graph of the solution reaches infinity (or negative infinity) in a finite time period.

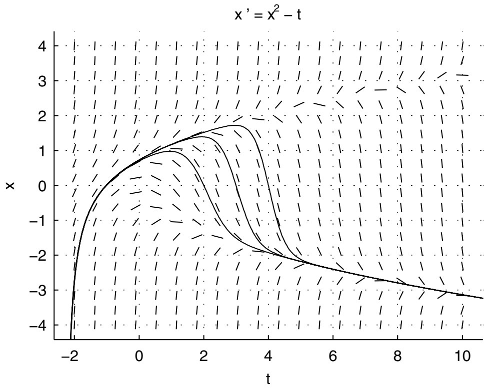

Figure 3.11. Do the trajectories intersect?

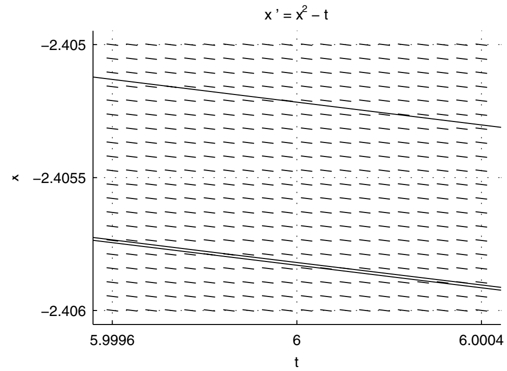

Figure 3.12. The trajectories don't merge or cross.

around the area of interest, release the mouse button and the contents of the zoom box will be magnified to the full size of the display window. The same effect can be achieved on a Macintosh with a one button mouse by depressing the option key while clicking and dragging.

`Dfield6` allows you to "zoom back" to revisit any previously used window. Select **Edit→Zoom back** in the DFIELD6 Display window. This will open the DFIELD6 Zoom back dialog box pictured in Figure 3.13. Select the window you wish to revisit and click the **Zoom** button.

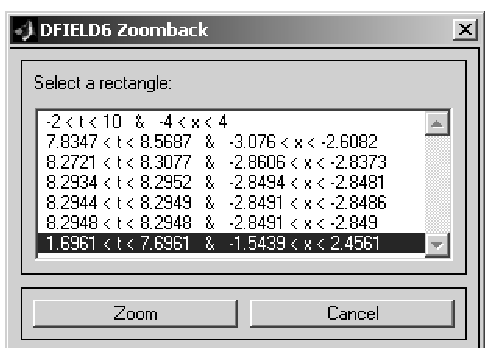

Figure 3.13. Select a zoom window and click the **Zoom** button.

Qualitative Analysis

Suppose that you model a population with a differential equation. If you want to use your model to predict the exact population in three years, then you will need to find an analytic or a numerical solution. However, if your only interest is what happens to the population after a long period of time, a qualitative approach might be easier and more appropriate.

Example 5. *Let $P(t)$ represent a population of bacteria at time t, measured in millions of bacteria.*

Suppose that P is governed by the logistic model

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right). \tag{3.3}$$

Assume that $r = 0.75$ and $K = 10$ and suppose that the initial population at time $t = 0$ is $P(0) = 1$. What will happen to this population over a long period of time?

Let's first examine the model experimentally using `dfield6`. Instead of filling out the DFIELD6 Setup window by hand, we can use the gallery by choosing **Gallery→logistic equation**. Notice that the *parameters* r and K have been given the correct values. To provide more room below $P = 0$, set the minimum value of P to be -4, as shown in Figure 3.14. Click the **Proceed** button to start the computation of the direction field.

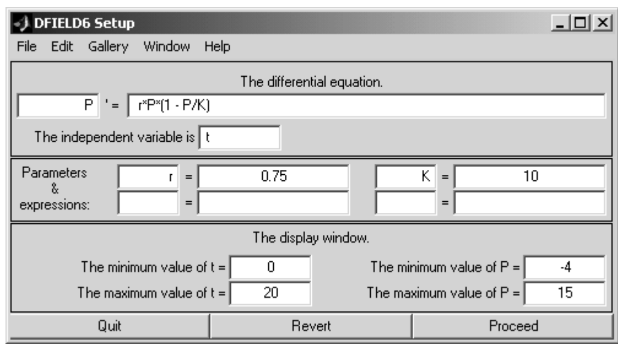

Figure 3.14. Setup window for $dP/dt = rP(1 - P/K)$.

Plot a few solutions by clicking the mouse at various points with $P > 0$. Notice that each of these solutions tends to 10 as t increases. Remember that $K = 10$. This is not a coincidence, and we will return to this point later. Some solution curves are shown in Figure 3.15.

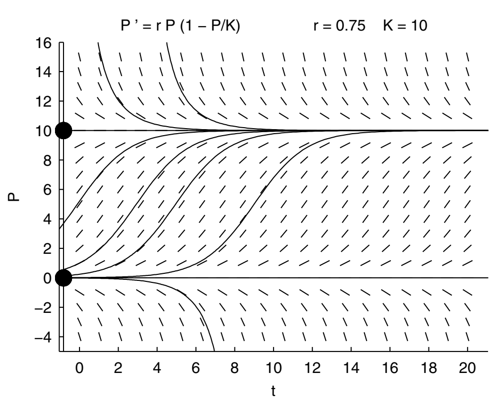

Figure 3.15. Solutions to $P' = rP(1 - P/K)$.

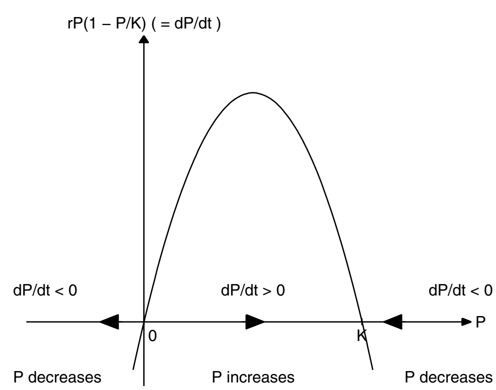

Figure 3.16. The plot of $rP(1 - P/K)$ versus P.

This behavior can be predicted quite easily using qualitative analysis. If you plot the right hand side of equation (3.3) versus P (i.e., plot $rP(1 - P/K)$ versus P), the result is the inverted parabola seen in Figure 3.16. Set $rP(1 - P/K)$ equal to zero to find that the graph crosses the P-axis in Figure 3.16 at $P = 0$ and $P = K$. These are called *equilibrium points*. It is easily verified that $P(t) = K$ is a solution of $dP/dt = rP(1 - P/K)$ by substituting $P(t) = K$ into each side of the differential equation and simplifying. Similarly, the solution $P(t) = 0$ is easily seen to satisfy the differential equation.

Although the solutions $P(t) = 0$ and $P(t) = K$ might be considered "trivial" since they are constant functions, they are by no means trivial in their importance. The solutions $P(t) = 0$ and $P(t) = K$ are called *equilibrium solutions*. For example, if $P(t) = K$, the growth rate dP/dt of the population is zero and the population remains at $P(t) = K$ forever. Similarly, if $P(t) = 0$, the growth rate dP/dt equals zero and the population remains at $P(t) = 0$ for all time.

When the graph of $rP(1 - P/K)$ (which is equal to dP/dt if P is a solution) falls below the P-axis in Figure 3.16, then $dP/dt < 0$ and the first derivative test implies that $P(t)$ is a decreasing function of t. On the other hand, when the graph of $rP(1 - P/K)$ rises above the P-axis in Figure 3.16, then $dP/dt > 0$ and $P(t)$ is an increasing function of t. These facts are summarized by the arrows on the P axis in Figure 3.16. This is an example of a *phase line*. The information on the phase line indicates that a population beginning between 0 and K million bacteria has to increase to the equilibrium value of K million bacteria. If the starting population is greater than K million then the population decreases to K million. For this reason the parameter K is called the *carrying capacity*.

Now let's go back to the DFIELD6 Display window. Select **Options→Keyboard input** and start solution trajectories with initial conditions $P(0) = 0$ and $P(0) = 10$. For the second trajectory you can enter $P = K$ instead of $P = 10$, since the use of parameters is allowed in the keyboard input window. In Figure 3.15, note that these equilibrium solutions are horizontal lines. Select **Options→ Solution direction→Forward** and **Options→Show the phase line** in the DFIELD6 Display window. `Dfield6` aligns the phase line from Figure 3.16 in a vertical direction at the left edge of the direction field in the DFIELD6 Display window. To see the motion along the phase line it is a good idea to slow the computations. Choose **Options→Solver settings** and move the slider to less than 10 solutions steps per second.

Next begin the solution with initial condition $P(0) = 1$ and note the action of the animated point on the phase line. As the solution trajectory in the direction field approaches the horizontal line $P = 10 = K$, the point on the phase line approaches equilibrium point $P = 10$ on the phase line, as shown in Figure 3.15. It would appear that a population with initial conditions and parameters described in the original problem statement will have to approach 10 million bacteria with the passage of time.

If you chose to slow the computation, perhaps you noticed something you had seen only fleetingly before. When a computation is started, a new button labelled **Stop** appears on the DFIELD6 Display window. If you click this button, the solution of the trajectory in the current direction is halted.

Experiment with some other initial conditions. Note that solutions beginning a little above or a little below the equilibrium solution $P = 10$ tend to move back toward this equilibrium solution with the passage of time. This is why the solution $P = 10$ is called an *assymptotically stable* equilibrium solution. However, solutions beginning a little above or a little below the equilibrium solution $P = 0$ tend to move away from this equilibrium solution with the passage of time. The solution $P = 0$ is called an *unstable* equilibrium solution. You can review the results of our experiments in Figure 3.15.

Using MATLAB While DFIELD6 is Open

All of the features of MATLAB are available while `dfield6` is open. You can use MATLAB commands to plot to the DFIELD6 Display window, or you can open another figure window by typing `figure` at the MATLAB prompt. When more than one figure is open, it is important to remember that plotting commands are directed to the *current figure*. This is always the most recently visited window. You can make a particular figure active by clicking on it. If the DFIELD6 Setup window is the current figure, your plot command will be directed to it. It will be executed correctly, but it will not change the appearance of the window, so it will look as though nothing happened. This is an annoying outcome. When you have more than one figure window open, it is a good idea to click on the figure where you want a plot to be executed just before issuing the command.

Example 6. *The behavior of a population is modeled by the logistic equation*

$$P' = r\,P\left(1 - \frac{P}{K}\right),$$

with $r = 1$. *However, in this case the carrying capacity is changing with time according to the equation* $K = K(t) = 3 + t$. *Use* `dfield6` *to plot several solutions. Plot the carrying capacity on the DFIELD6 Display window to facilitate comparison between the long-term behavior of the solutions and the carrying capacity.*

First we enter the equation into the DFIELD6 Setup window as in Figure 3.17. The only new feature here is that we entered the carrying capacity `K = 3 + t` as an expression. Any mathematical expression involving the dependent and independent variables can be entered.

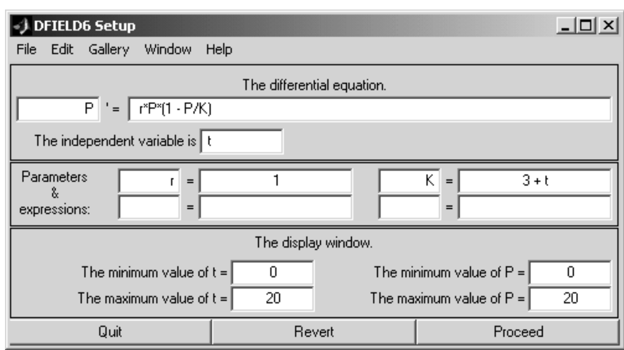

Figure 3.17. The setup window for the equation in Example 6.

Next we proceed to the DFIELD6 Display window and plot a few solutions (see Figure 3.18). Notice that ultimately all solutions seem to merge together and increase linearly, as does the carrying capacity. To see the relationship between the limiting behavior of the solutions and the carrying capacity more clearly, we use the commands

```
t = linspace(-2,22);
plot(t,3+t,'r')
```

to plot the carrying capacity in red. We clicked on the DFIELD6 Display window just before executing the `plot` command to make sure that it is the current figure. The result is shown in Figure 3.18, where we thickened the graph of the carrying capacity since we are not able to show a red curve in this manual. This can be done with the command `plot(t,3+t,'linewidth',2)`. You will notice that the limiting behavior of the solutions is linear growth, parallel to the graph of the carrying capacity.

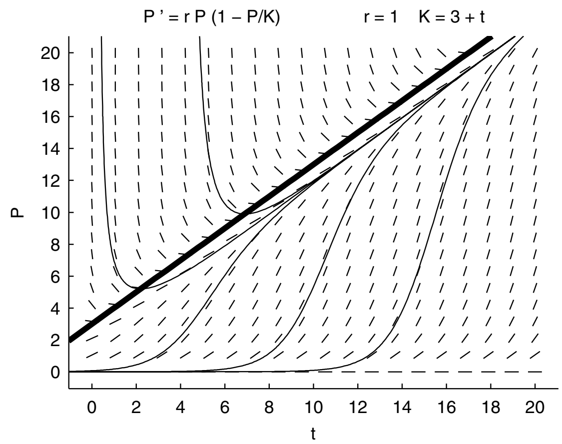

Figure 3.18. Plotting in the DFIELD6 Display window.

Subscripts and Greek Letters

The voltage V_c on the capacitor in an RC-circuit satisfies the differential equation

$$RCV_c' + V_c = A\cos\omega t, \tag{3.4}$$

where R is the resistance in ohms, C is the capacitance in farads, and $A\cos\omega t$ is a sinusoidal external voltage with amplitude A and frequency ω.

Example 7. *Use* `dfield6` *to study the response of an RC-circuit to external voltages of different frequencies. Use $R = 0.05\Omega$ and $C = 2F$. For $A = 5$ and $\omega = 1,\ 2,\ 5,\ 10,\ 20,\ 50,$ and 100 find the amplitude of the steady-state response with $V_c(0) = 0$. Why do you think an RC circuit is sometimes called a low-pass filter?*

Entering this equation into the DFIELD6 Setup window is easy since it is in the gallery. Choose **Gallery→RC circuit**, and make the needed change $R = 0.05$. The result is Figure 3.19. Now it is only necessary to change the input for the parameter ω and, when necessary, the maximum value of t to complete the exercise. In doing so you will notice that low frequency voltages pass through the RC circuit with their amplitudes practically unchanged, while high frequencies are attenuated. Hence the name *low-pass filter.* The results for two frequencies are shown in Figures 3.20 and 3.21.

Notice that the subscripted voltage V_c is entered as `V_c` into the DFIELD6 Setup window, and appears nicely subscripted in the DFIELD6 Display window. This is an example of TEX (or LATEX) notation. If you want a subscripted quantity to appear on a MATLAB figure window it is only necessary

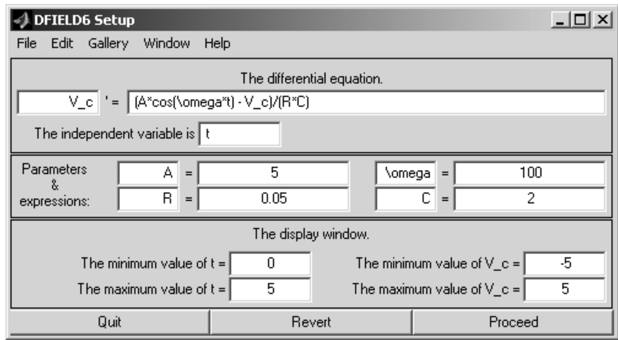

Figure 3.19. The setup window for Example 7.

to precede the subscript by an underscore. If the subscript contains more than one letter, put the entire subscript between curly brackets (`{}`). If you have a superscripted quantity, precede the superscript with a caret (`^`). Finally, notice that the frequency ω is entered in the setup window as `\omega` and appears in the display window in its proper Greek form. This, too, is TEX notation. Most Greek letters, including some upper case letters, can be treated this way. Simply spell out the name, preceded by a backslash. For example, you can use `\alpha`, `\beta`, `\gamma`, `\theta`, `\phi`, `\Delta`, `\Omega`, and `\Theta`.

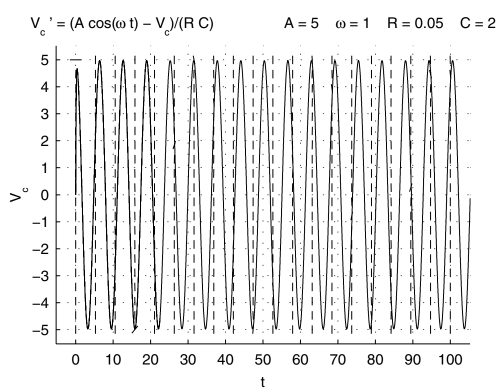

Figure 3.20. Response for $\omega = 1$.

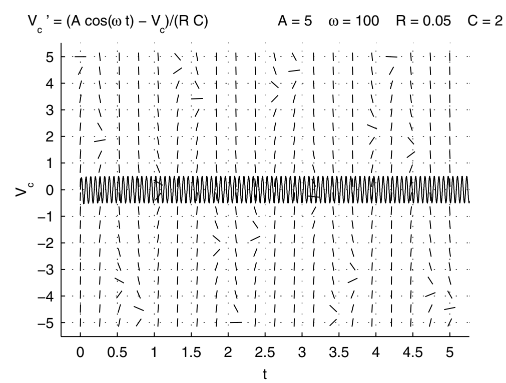

Figure 3.21. Response for $\omega = 100$.

Editing the Display Window

The appearance of the Display Window can be changed in a variety of ways.

Changing window settings. The menu item **Options→Windows settings** provides several ways to alter the appearance of the DFIELD6 Display window. Selecting this item will open the DFIELD6 Windows settings dialog box (see Figure 3.22).

The first option involves the three radio buttons, and allows you to choose between a line field, a vector field, or no field at all. Some people prefer to use a *vector field* to a direction line field. In a vector

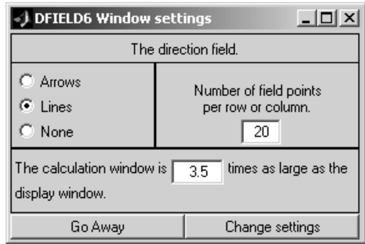

Figure 3.22. DFIELD6 Window settings.

field, a vector is attached to each point instead of the line segment used in a direction field. The vector has its base at the point in question, its direction is the slope, and the length of the vector reflects the magnitude of the derivative. Click the **Change settings** button to make any change you select.

There is an edit box in the DFIELD6 Window settings dialog that allows the user to choose the number of field points displayed. The default is 20 points in each row and in each column. Change this entry to 10, hit **Enter**, then click the **Change settings** button to note the affect on the direction field.

The design of `dfield6` includes the definition of two windows: the DFIELD6 Display window and the *calculation window*. When you start `dfield6`, the calculation window is 3.5 times as large as the display window in each dimension. The computation of a solution will stop only when the solution curve leaves the calculation window. This allows some room for zooming to a larger display window without having incomplete solution curves. It also allows for some reentrant solution curves — those which leave the display window and later return to it. The third item in the DFIELD6 Window settings dialog box controls the relative size of the calculation window. It can be given any value greater than or equal to 1. The smaller this number the more likely that reentrant solutions will be lost. The default value of 3.5 seems to meet most needs, but if you are losing too many reentrant solutions you can increase this parameter.

Marking initial points. It is possible to mark the points at which the computation of solutions is started. To do this, select **Options→Mark initial points**. To stop doing so, select the same option to uncheck it. Initial points that are already plotted can be erased with the command **Edit→Erase all marked initial points**.

Level curves. Sometimes it is useful to plot level sets of functions in the DFIELD6 Display window. In Example 6, instead of plotting the curve $P = K = 3 + t$ from the command line we could have plotted the level curve $P - K = 0$. This can be done using the command **Options→Plot level curves**. Complete the window as shown in Figure 3.23 and click Proceed. If you want to remove the level sets, select **Edit→Erase all level curves**.

Erasing objects. Sometimes when you are preparing a display window for printing, you plot a solution curve you wish were not there. In the **Edit** menu there are several commands which allow you to erase items in the DFIELD6 Display window. In addition to those we have already explained, there are **Edit→ Erase all solutions** and **Edit→Erase all graphics objects**, which are self explanatory. The last item, **Edit→Delete a graphics object**, is much more flexible. It will allow you to delete individual solution curves, as well as text items and graphs you have added to the window. Simply choose the option and select the object you wish to delete with the mouse.

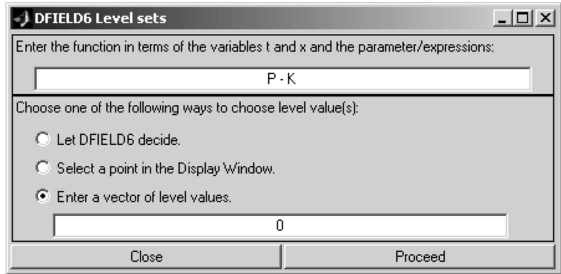

Figure 3.23. The DFIELD6 Level sets window.

Text objects in DFIELD6. The DFIELD6 Display window is a standard MATLAB figure window. Therefore all of the standard editing commands which we described in Chapter 2 are available. In particular the commands `xlabel`, `ylabel`, and `title` can be used to change these items. To add text at arbitrary points in the DFIELD6 Display window, select **Edit→Enter text on the Display Window**, enter the desired text in the Text entry dialog box, and then click the **OK** button. Use the mouse to click at the lower left point of the position in the figure window where you want the text to appear. It can easily happen that your placement of the text does not please you. If so, remove the text using **Edit→Delete a graphics object**, then try again.

Using the Property Editors with DFIELD6. We described the use of the Property Editors in Chapter 2. While these methods are very powerful and not difficult to use, they do not mix flawlessly with the interactive aspects of `dfield6`. You should be careful when you use them with the DFIELD6 Display window. It is a good idea to complete all of your `dfield6` work first, and only then begin to use the formatting commands. Do not mix them. It is not unusual that MATLAB freezes when the two are mixed. It is usually a good idea to select **Options→Make the display window inactive** before using the tools in the toolbar.

Other Features of DFIELD6

Printing, Saving, and Using Clipboards. You can print or export the DFIELD6 Display window in any of the ways described in Chapter 2. However, the easiest way to print the figure to the default printer is to click the **Print** button in the DFIELD6 Display window. The **Print** and **Quit** buttons and the message window will not be printed.

Saving and Loading DFIELD6 Equation and Gallery Files. Suppose that after entering all of the information into the DFIELD6 Setup window for Example 7, as it appears in figure 3.19, you decided to work on something else and come back to this example later. There are two ways to avoid the necessity of reentering the data. The first method is temporary. The menu option **Gallery→Add current equation to the gallery** will do just that, after prompting you for a name for the equation. When you are ready, you can choose this equation from the **Gallery** menu, and all of the data will be entered automatically.

However, if you have to quit `dfield6`, the new entry will no longer be there when you come back. For this situation you can use the command **File→Save the current equation ....** This option allows you to record the information on the DFIELD6 Setup window in a file. Executing this option will bring up a standard file save menu, where you are given the option of saving the file with a filename and in a directory of your own choice. The file will be saved with the suffix `.dfs`. (It is not necessary to enter `.dfs`.) It can later be loaded back into `dfield6` using the command **File→Load an equation ...**.

It is also possible to save and load entire galleries using the appropriate commands on the **File** menu. Gallery files have the suffix `.dfg`. There is also a command that will delete the entire gallery, allowing you to start to build a gallery entirely your own, and another command that will reload the default gallery, if that is what you want.

Quitting DFIELD6. Always wait until the word "Ready" appears in the `dfield6` message window before you try to do anything else with `dfield6` or MATLAB. When you want to quit `dfield6`, the best way is to use the **Quit** buttons found on the DFIELD6 Setup or on the DFIELD6 Display windows. Either of these will close all of the `dfield6` windows in an orderly manner, and it will delete the temporary files that `dfield6` creates in order to do its business.

Plotting Several Solutions at Once. `Dfield6` allows you to plot several solutions at once. Select **Options→Plot several solutions** and note that the mouse cursor changes to "cross-hairs" when positioned over the direction field. Select several initial conditions for your solutions by clicking the mouse button at several different locations in the direction field. When you are finished selecting initial conditions, position the mouse cross-hairs over the direction field and press the **Enter** or **Return** key on your keyboard. Solution trajectories will emanate from the initial conditions you selected with the mouse.

Exercises

For the differential equations in Exercises 1–4, perform each of the following tasks.

a) Print out the direction field for the differential equation with the display window defined by $t \in [-5, 5]$ and $y \in [-5, 5]$. You might consider increasing the number of field points to 25 in the DFIELD6 Window settings dialog box. On this printout, sketch with a pencil as best you can the solution curves through the initial points $(t_0, y_0) = (0, 0)$, $(-2, 0)$, $(-3, 0)$, $(0, 1)$, and $(4, 0)$. Remember that the solution curves must be tangent to the direction lines at each point.

b) Use `dfield6` to plot the same solution curves to check your accuracy. Turn in both versions.

1. $y' = ty$.
2. $y' = y^2 - t^2$.
3. $y' = 2ty/(1 + y^2)$.
4. $y' = y(2 + y)(2 - y)$.
5. Use `dfield6` to plot a few solution curves to the equation $x' + x\sin(t) = \cos(t)$. Use the display window defined by $x \in (-10, 10)$ and $t \in (-10, 10)$.
6. Use `dfield6` to plot the solution curves for the equation $x' = 1 - t^2 + \sin(tx)$ with initial values $x = -3, -2, -1, 0, 1, 2, 3$ at $t = 0$. Find a display window which shows the most important features of the solutions by experimentation.

For the differential equations in Exercises 7–10 perform the following tasks.

a) Use `dfield6` to plot a few solutions with different initial points. Start with the display window bounded by $0 \le t \le 10$ and $-5 \le y \le 5$, and modify it to suit the problem. Print out the display window and turn it in as part of this assignment.

b) Make a conjecture about the limiting behavior of the solutions of as $t \to \infty$.

c) Find the general analytic solution to this equation.

d) Verify the conjecture you made in part b), or if you no longer believe it, make a new conjecture and verify that.

7. $y' + 4y = 8$.
8. $(1 + t^2)y' + 4ty = t$.
9. $ty' + ty = 2 - y$.
10. $(1 + t)y' = y(4 - y^2)$.

In Exercises 11–14 we will consider a certain lake which has a volume of $V = 100$ km^3. It is fed by a river at a rate of r_i km^3/year, and there is another river which is fed by the lake at a rate which keeps the volume of the lake constant. In addition, there is a factory on the lake which introduces a pollutant into the lake at the rate of p km^3/year. This means that the rate of flow from the lake into the outlet river is $(p + r_i)$ km^3/year. Let $x(t)$ denote the volume of the pollutant in the lake at time t, and let $c(t) = x(t)/V$ denote the concentration of the pollutant.

11. Show that, under the assumption of immediate and perfect mixing of the pollutant into the lake water, the concentration satisfies the differential equation $c' + ((p + r_i)/V)c = p/V$.
12. Suppose that $r_i = 50$, and $p = 2$.
    a) Assume that the factory starts operating at time $t = 0$, so that the initial concentration is 0. Use `dfield6` to plot the solution. Remember the definition of the concentration is x/V so you can be sure it is pretty small. Choose the dimensions of the display window carefully.
    b) It has been determined that a concentration of over 2% is hazardous for the fish in the lake. Approximately how long will it take until this concentration is reached? You can "zoom in" on the `dfield6` plot to enable a more accurate estimate.
    c) What is the limiting concentration? About how long does it take for the concentration to reach a concentration of 3.5%?
13. Suppose the factory stops operating at time $t = 0$, and that the concentration was 3.5% at that time. Approximately how long will it take before the concentration falls below 2%, and the lake is no longer hazardous for fish? Notice that $p = 0$ for this exercise.
14. Rivers do not flow at the same rate the year around. They tend to be full in the Spring when the snow melts, and to flow more slowly in the Fall. To take this into account, suppose the flow of our river is

$$r_i = 50 + 20\cos(2\pi(t - 1/3)).$$

    Our river flows at its maximum rate one-third into the year, i.e., around the first of April, and at its minimum around the first of October.
    a) Setting $p = 2$, and using this flow rate, use `dfield6` to plot the concentration for several choices of initial concentration between 0% and 4%. (If your solution seems erratic, reduce the relative error tolerance using **Options→Solver settings**.) How would you describe the behavior of the concentration for large values of time?
    b) It might be expected that after settling into a steady state, the concentration would be greatest when the flow was smallest, around the first of October. At what time of the year does the highest concentration actually occur? Reduce the error tolerance until you get a solution curve smooth enough to make an estimate.
15. Use `dfield6` to plot several solutions to the equation $z' = (z - t)^{5/3}$. (**Hint:** Notice that when $z < t$, $z' < 0$, so the direction field should point down, and the solution curves should be decreasing. You might have difficulty getting the direction field and the solutions to look like that. If so read the section in Chapter 1 on complex arithmetic, especially the last couple of paragraphs.)

A differential equation of the form $dx/dt = f(x)$, whose right-hand side does not explicitly depend on the independent variable t, is called an *autonomous* differential equation. For example, the logistic model in Example 5 was autonomous. For the autonomous differential equations in Exercises 16 – 19, perform each of the following tasks. Note that the first three tasks are to be performed without the aid of technology.

a) Set the right-hand side of the differential equation equal to zero and solve for the equilibrium points.
b) Plot the graph of the right-hand side of each autonomous differential equation versus x, as in Figure 3.16. Draw the phase line below the graph and indicate where x is increasing or decreasing, as was done in Figure 3.16.

c) Use the information in parts a) and b) to draw sample solutions in the xt plane. Be sure to include the equilibrium solutions.

d) Check your results with `dfield6`. Again, be sure to include the equilibrium solutions.

e) If x_0 is an equilibrium point, i.e., if $f(x_0) = 0$, then $x(t) = x_0$ is an equilibrium solution. It can be shown that if $f'(x_0) < 0$, then every solution curve that has an initial value near x_0 converges to x_0 as $t \to \infty$. In this case x_0 is called a *stable* equilibrium point. If $f'(x_0) > 0$, then every solution curve that has an initial value near x_0 diverges away from x_0 as $t \to \infty$, and x_0 is called an *unstable* equilibrium point. If $f'(x_0) = 0$, no conclusion can be drawn about the behavior of solution curves. In this case the equilibrium point may fail to be either stable or unstable. Apply this test to each of the equilibrium points.

16. $x' = \cos(\pi x)$, $x \in [-3, 3]$.
17. $x' = x(x-2)$, $-\infty < x < \infty$.
18. $x' = x(x-2)^2$, $-\infty < x < \infty$.
19. $x' = x(x-2)^3$, $-\infty < x < \infty$.

In Exercises 20 – 22 you will not be able to solve explicitly for all of the equilibrium points. Instead, turn the problem around. Use `dfield6` to plot some solutions, and from that information calculate approximately where the equilibrium points are, and determine the type of each. In Exercise 20 you can check your estimate with the code:

```
f=inline('x*(1+exp(-x)-x^2)')
z=fzero(f,1)
f(z)
```

Similar methods will help for Exercises 21 and 22.

20. $x' = x(1 + e^{-x} - x^2)$, $-1 \le x \le 2$.
21. $x' = x^3 - 3x + 1$.
22. $x' = \cos x - 2x$.

The logistic equation $P' = r\,P(1 - P/K)$ is discussed in Examples 5 and 6. Usually the parameter's r and K are constants and in Example 5 we found that for any solution $P(t)$ which has a positive initial value we have $P(t) \to K$ as $t \to \infty$. For this reason K is called the *carrying capacity* of the system. However, in Example 6 we saw a case where the carrying capacity is not constant, yet we were able to show how the limiting behavior of the population related to the carrying capacity. In Exercises 23–26 you are to examine the long term behavior of solutions, especially in comparison to the carrying capacity. In particular:

a) Use `dfield6` to plot several solutions to the equation. (It is up to you to find a display window that is appropriate to the problem at hand.)

b) Based on the plot done in part a), describe the long term behavior of the solutions to the equation. In particular, compare this long term behavior to that of K. It might be helpful to plot K on the display window as we did in Example 6. In the first case the solutions will all be asymptotic to a constant. In the other two the solutions will all have the same long term behavior. Describe that behavior in comparison to the graph of K. The results of Examples 5 and 6 should be helpful.

23. $K(t) = 1 - \frac{1}{2}e^{-t}$, $r = 1$. In this case $K(t)$ is monotone increasing, and $K(t)$ is asymptotic to 1. This might model a situation of a human population where, due to technological improvement, the availability of resources is increasing with time, although ultimately limited.
24. $K(t) = \sqrt{1+t}$, $r = 1$. Again $K(t)$ is monotone increasing, but this time it is unbounded. This might model a situation of a human population where, due to technological improvement, the availability of resources is steadily increasing with time, and therefore the effects of competition are becoming less severe.
25. $K(t) = 1 - \frac{1}{2}\cos(2\pi t)$, $r = 1$. This is perhaps the most interesting case. Here the carrying capacity is periodic in time with period 1, which should be considered to be one year. This models a population of insects or small animals that are affected by the seasons. You will notice that the long term behavior as $t \to \infty$ reflects the behavior of K. The solution does not tend to a constant, but nevertheless all solutions have the same long term behavior for large values of t. In particular, you should take notice of the location of the

maxima and minima of K and of P and how they are related. You can use the "zoom in" option to get a better picture of this.

26. $K(t) = \sqrt{1+t} - \frac{1}{2}\cos(2\pi t)$, $r = 1$.

27. Despite the seeming generality of the uniqueness theorem, there are initial value problems which have more than one solution. Consider the differential equation $y' = \sqrt{|y|}$. Notice that $y(t) \equiv 0$ is a solution with the initial condition $y(0) = 0$. (Of course by $\sqrt{|y|}$ we mean the **nonnegative** square root.)

    a) This equation is separable. Use this to find a solution to the equation with the initial value $y(t_0) = 0$ assuming that $y \geq 0$. You should get the answer $y(t) = (t - t_0)^2/4$. Notice, however, that this is a solution only for $t \geq t_0$. Why?

    b) Show that the function

    $$y(t) = \begin{cases} 0, & \text{if } t < t_0; \\ (t-t_0)^2/4, & \text{if } t \geq t_0; \end{cases}$$

    is continuous, has a continuous first derivative, and satisfies the differential equation $y' = \sqrt{|y|}$.

    c) For any $t_0 \geq 0$ the function defined in part b) satisfies the initial condition $y(0) = 0$. Why doesn't this violate the uniqueness part of the theorem?

    d) Find another solution to the initial value problem in a) by assuming that $y \leq 0$.

    e) You might be curious (as were the authors) about what `dfield6` will do with this equation. Find out. Use the rectangle defined by $-1 \leq t \leq 1$ and $-1 \leq y \leq 1$ and plot the solution of $y' = \sqrt{|y|}$ with initial value $y(0) = 0$. Also, plot the solution for $y(0) = 10^{-50}$ (the MATLAB notation for 10^{-50} is `1e-50`). Plot a few other solutions as well. Do you see evidence of the non-uniqueness observed in part c)?

An important aspect of differential equations is the dependence of solutions on initial conditions. There are two points to be made. First, we have a theorem which says that the solutions are continuous with respect to the initial conditions. More precisely,

Theorem. *Suppose that the function $f(t,x)$ is defined in the rectangle R defined by $a \leq t \leq b$ and $c \leq x \leq d$. Suppose also that f and $\partial f/\partial x$ are both continuous in R, and that*

$$\left|\frac{\partial f}{\partial x}\right| \leq L \quad \text{for all } (t,x) \in R.$$

If (t_0, x_0) and (t_0, y_0) are both in R, and if

$$\begin{aligned} x' &= f(t,x) \\ x(t_0) &= x_0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} y' &= f(t,y) \\ y(t_0) &= y_0 \end{aligned}$$

then for $t > t_0$

$$|x(t) - y(t)| \leq e^{L(t-t_0)}|x_0 - y_0|$$

as long as both solution curves remain in R.

Roughly, the theorem says that if we have initial values that are sufficiently close to each other, the solutions will remain close, at least if we restrict our view to the rectangle R. Since it is easy to make measurement mistakes, and thereby get initial values off by a little, this is reassuring.

For the second point, we notice that although the dependence on the initial condition is continuous, the term $e^{L(t-t_0)}$ allows the solutions to get exponentially far apart as the interval between t and t_0 increases. That is, the solutions can still be extremely sensitive to the initial conditions, especially over long t intervals.

28. Consider the differential equation $x' = x(1 - x^2)$.

    a) Verify that $x(t) \equiv 0$ is the solution with initial value $x(0) = 0$.

b) Use `dfield6` to find approximately how close the initial value y_0 must be to 0 so that the solution $y(t)$ of our equation with that initial value satisfies $y(t) \le 0.1$ for $0 \le t \le t_f$, with $t_f = 2$. You can use the display window $0 \le t \le 2$, and $0 \le x \le 0.1$, and experiment with initial values in the **Options→ Keyboard input** window, until you get close enough. Do not try to be too precise. Two significant figures is sufficient.

c) As the length of the t interval is increased, how close must y_0 be to 0 in order to insure the same accuracy? To find out, repeat part b) with $t_f = 4$, 6, 8, and 10.

The results of the last problem show that the solutions can be extremely sensitive to changes in the initial conditions. This sensitivity allows chaos to occur in deterministic systems, which is the subject of much current research.

One way to experience first hand the sensitivity to changes in the initial conditions is to try a little "target practice." For the ODEs in Exercises 29–33, use `dfield6` to find approximately the value of x_0 such that the solution $x(t)$ to the initial value problem with initial condition $x(0) = x_0$ satisfies $x(t_1) = x_1$. You should use the Keyboard input window to initiate the solution. Widen the window to allow a large number of digits in the edit window by clicking and dragging on the right edge. After an unsuccessful attempt try again with another initial condition. The Uniqueness Theorem should help you limit your choices. If you make sure that the Display Window is the current figure (by clicking on it), and execute `plot(t1,x1,'or')` at the command line, you will have a nice target to shoot at.

You will find that hitting the target gets more difficult in each of these problems. We allow you to "cheat" by starting a solution in the target, and finding the value at $t = 0$. However, be sure to try to hit the target with that initial value. You may be surprised at the outcome.

29. $x' = x - \sin(x)$, $t_1 = 5$, $x_1 = 2$.
30. $x' = x^2 - t$, $t_1 = 4$, $x_1 = 0$.
31. $x' = x(1 - x^2)$, $t_1 = 5$, $x_1 = 0.5$.
32. $x' = x\sin(x) + t$, $t_1 = 5$, $x_1 = 0$.
33. $x' = x\sin(x^2) + t$, $t_1 = 5$, $x_1 = 1$. In this case the authors were not able to hit the target. However, the exercise of trying is still worthwhile. We leave it to you to ponder why it is not possible.

Appendix: Downloading and Installing the Software Used in This Manual

The MATLAB programs `dfield`, `pplane`, and `odesolve`, and the solvers `eul`, `rk2`, and `rk4` described in this manual are not distributed with MATLAB. They are MATLAB function M-files and are available for download over the internet. There are versions of `dfield` and `pplane` written for use with all recent versions of MATLAB. However, `odesolve` is new, and only works with MATLAB ver 6.0 and later. The solvers are the same for all versions of MATLAB.

The following three step procedure will insure a correct installation, but the only important point is that the files must be saved as MATLAB M-files in a directory on the MATLAB path.

- Create a new directory with the name `odetools` (or choose a name of your own). It can be located anywhere on your directory tree, but put it somewhere where you can find it later.
- In your browser, go to `http://math.rice.edu/~dfield/`. For each file you wish to download, click on the link. In Internet Explorer, you are given the option to save the file. In Netscape, the file for the software will open in your browser, and you can save the file using the File menu. In either case, save the file with the subscript `.m` in the directory `odetools`.
- Open the path tool by executing the command `pathtool` in the MATLAB command window, or by selecting **File→Set Path ...**. Follow the instructions for adding the directory `odetools` to the path. If you are asked if you want the change to be permanent, say yes.

From this point on, the programs will be available in MATLAB.