

Math 211
Final Exam

December 10, 1997

Part 2

Instructions: Write out and sign the honor pledge on your exam paper for Part 2. In addition put the name of your instructor in a prominent place on your exam. It is due by 3:30 PM on Saturday, December 20, in the Mathematics Department Office, HB 220.

Part 1 is worth 120 points, and Part 2 is worth 80 points.

Part 2 of the exam is an open book, open notes, take home exam. While the exam emphasizes the computer aspect of the course, you are allowed to use any of the analytic methods discussed in the course. In fact you will be required to.

You will be required to use `ode45` on this exam. You will remember that the new version of `ode45` requires that your derivative m-file should output a column vector. If you do not remember how to do that, you may ask a friend, or one of the instructors.

There are no limits on what MATLAB functions you can use.

The first part of the exam deals with the planar system

$$\begin{aligned}x' &= ax + y - x(x^2 + y^2) \\y' &= -x + ay - y(x^2 + y^2)\end{aligned}$$

for the three cases when $a = -1/4$, $a = 0$, and $a = 1/4$. You are to discover the nature of the solutions to the system as $t \rightarrow \infty$. It is suggested that you use `pplane5` to examine these systems in order to get insight into the questions we ask. It will be most useful if you use the rectangle $-1 \leq x \leq 1$, $-1 \leq y \leq 1$. It is not necessary to submit any plots. Do so only if you see something that looks really important.

You will also be asked how the function $F(x, y) = x^2 + y^2$ varies along solution curves. Notice that the level set $F(x, y) = x^2 + y^2 = r^2$ is the circle of radius r with center at the origin.

Answer: We can compute the Jacobian at the origin for general values of a :

$$J(0, 0) = \begin{pmatrix} a & 1 \\ -1 & a \end{pmatrix},$$

which has eigenvalues $a \pm i$. To examine how F varies along solution curves we compute

$$\begin{aligned}\frac{d}{dt}F(x(t), y(t)) &= 2x(ax + y - x(x^2 + y^2)) + 2y(-x + ay - y(x^2 + y^2)) \\ &= 2(x^2 + y^2)(a - (x^2 + y^2))\end{aligned}$$

1. (7 points) For the case $a = -1/4$, describe the behavior of the solutions as $t \rightarrow \infty$. Your description should include the classification of the equilibrium point at the origin, and the behavior of F along the solution curves.

Answer: With $a = -1/4$, the eigenvalues are $-1/4 \pm i$. Consequently the origin is a spiral sink. In addition we see that

$$\frac{d}{dt}F(x(t), y(t)) = 2(x^2 + y^2)(-1/4 - (x^2 + y^2)) \leq 0.$$

Hence F is a Liapounov function on the whole plane, and we see that a solution starting anywhere in the plane spirals into the origin as $t \rightarrow \infty$.

2. (7 points) Repeat Problem 1 for the case $a = 0$. Does your analysis agree with what pplane5 tells you? Which do you believe, your analysis or pplane5?

Answer: In this case the eigenvalues are $\pm i$. Hence we can only say that the equilibrium point at the origin has spiral behavior. However, we have

$$\frac{d}{dt}F(x(t), y(t)) = -2(x^2 + y^2)^2 \leq 0,$$

so again F is a Liapounov function on the whole plane, and we see that a solution starting anywhere in the plane spirals into the origin as $t \rightarrow \infty$.

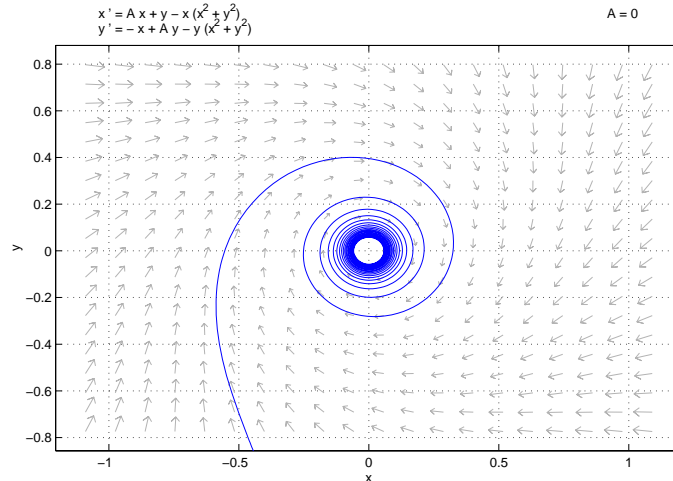


Figure 1. pplane5 for Problem 2.

Figure 1 shows a typical solution curve plotted by pplane5. The solution seems to end well short of the origin. The analysis of the previous paragraph shows that this does not happen. It is time to believe the analysis and not pplane5. (It is not necessary to provide Figure 1 to get full credit.)

3. (7 points) Repeat Problem 1 for the case $a = 1/4$. Can you find an exact solution along which F is a conserved quantity?

Answer: Now the eigenvalues are $1/4 \pm i$, so the origin is a spiral source. Analysis of F shows that

$$\frac{d}{dt}F(x(t), y(t)) = 2(x^2 + y^2)(1/4 - (x^2 + y^2)).$$

Hence

$$\frac{d}{dt}F(x(t), y(t)) \begin{cases} < 0, & \text{if } x^2 + y^2 > 1/4; \\ > 0, & \text{if } x^2 + y^2 < 1/4; \\ = 0, & \text{if } x^2 + y^2 = 1/4; \end{cases}$$

Thus F is a Liapounov function on the portion of the plane outside the circle of radius $1/2$ centered at the origin, and $-F$ is a Liapounov function inside the circle. Thus a solution starting outside the circle is drawn into the circle, and a solutions starting inside the circle is drawn out.

It appears that the circle itself is a solition curve. Is this true? Well on the circle the differential equations become

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned}$$

It is easy to see that

$$\begin{aligned} x(t) &= \sin(t)/2 \\ y(t) &= \cos(t)/2 \end{aligned}$$

solves these equations and the original system as well. For this solution the solution curve is the circle of radius $1/2$, and along this circle the function F is a conserved quantity.

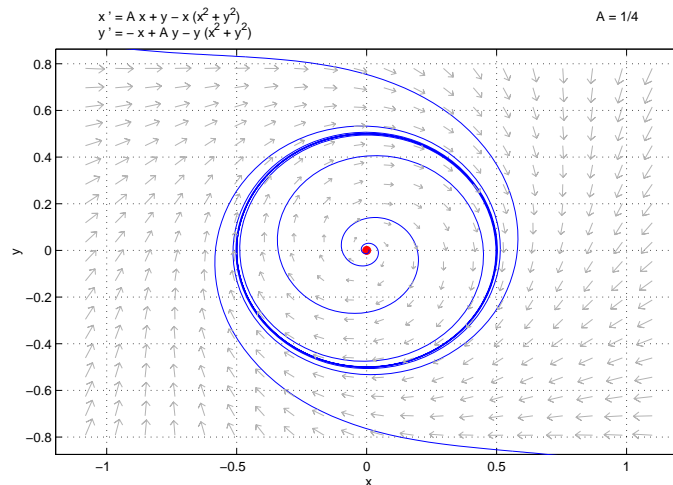


Figure 2. The limit cycle in Problem 3.

If we look at the results of pplane5 for this system as shown in Figure 2, we see these facts illustrated. The figure illustrates what is called a limit cycle, in this case the circle of radius 1/2. Neither Figure 1 nor reference to a limit cycle is needed for full credit on this problem.

The second part of the exam examines a predator prey situation. In our problem there are one predator and two prey populations. We will denote the predator population by $p(t)$, and the preys by $q(t)$ and $r(t)$. The distinguishing feature of our situation is that the predator strongly prefers prey q to prey r , and if there is a choice will feast on q instead of on r .

4. (7 points) Of the following three systems, which best models the situation described in the previous paragraph? Give reasons for your answer. (b is a positive constant.)

1)

$$\begin{aligned} p' &= p(-1 + q - re^{-q}) \\ q' &= q(1 - 2p) \\ r' &= r(1 + bpe^{-q}) \end{aligned}$$

2)

$$\begin{aligned} p' &= p(-1 + q + re^{-q}) \\ q' &= q(1 - 2p) \\ r' &= r(1 - bpe^{-q}) \end{aligned}$$

3)

$$\begin{aligned} p' &= p(-1 + q - (1 - q)r) \\ q' &= q(1 - 2p) \\ r' &= r(1 - b(1 + q)p) \end{aligned}$$

Answer: From the description we would expect that an increase in the prey r would result in an increased reproduction rate for the predator. In systems 1) & 3) the opposite happens, so neither of these systems is a good candidate. On the other hand system 2) has this property, and in addition the exponential term indicates that there is less pressure on r as q increases. This too is in agreement with the description.

In what follows we will consider the system

(*)

$$\begin{aligned} p' &= p(-1 + q + re^{-q}) \\ q' &= q(1 - 2p) \\ r' &= r(1 - bpe^{-q}) \end{aligned}$$

for a couple of values of the parameter b .

5. (8 points) Show that the Jacobian (i.e., the matrix of the linearization) of system (*) is

$$J = \begin{pmatrix} -1 + q + re^{-q} & p(1 - re^{-q}) & pe^{-q} \\ -2q & 1 - 2p & 0 \\ -bre^{-q} & brpe^{-q} & 1 - bpe^{-q} \end{pmatrix}$$

Answer: This is simple differentiation.

6. (11 points) Consider system (*) when $r = 0$.

- a) Show that the system reduces to a system of dimension 2, which we discussed at some length during the semester. In one sentence identify that system.

Answer: When $r = 0$, the third equation in system (*) is $r' = 0$. Hence r is constant and equal to 0 along the solution curves and the system becomes

$$\begin{aligned} p' &= p(-1 + q) \\ q' &= q(1 - 2p) \end{aligned}$$

This is a standard predator-prey system.

- b) The two dimensional system in part a) has an equilibrium point (p_1, q_1) with both components positive. Find it.

Answer: The equilibrium point is $(1/2, 1)$.

- c) Show that $(p_1, q_1, 0)$ is an equilibrium point for system (*). For the two cases $b = 1$ and $b = 3$, use the Jacobian to try to decide whether or not the equilibrium point is a sink. There are three acceptable answers: “Yes, it is a sink,” “No, it is not a sink,” and “It is impossible to decide with the information available.” Give a reason for your answer.

Answer: Direct substitution shows that $(1/2, 1, 0)$ is an equilibrium point. For $r = 0$ the Jacobian becomes

$$J = \begin{pmatrix} -1 + q & p & pe^{-q} \\ -2q & 1 - 2p & 0 \\ 0 & 0 & 1 - bpe^{-q} \end{pmatrix}$$

and with $p = 1/2$ and $q = 1$, this becomes

$$J = \begin{pmatrix} 0 & 1/2 & e^{-1}/2 \\ -2 & 0 & 0 \\ 0 & 0 & 1 - be^{-1}/2 \end{pmatrix}$$

It is not difficult to discover that the eigenvalues are $\pm i$ and $1 - b/2e$. Thus when $b < 2e$ there is a positive eigenvalue. This is true for both $b = 1$ and $b = 3$, so in both cases we know that the equilibrium point is not a sink.

It would be quite natural for students to do this computation in MATLAB. There are a variety of ways to proceed.

7. (11 points) Repeat Problem 6 when $q = 0$ instead of $r = 0$. We will label the equilibrium point here (p_2, r_2) , and then $(p_2, 0, r_2)$ should be an equilibrium point for system (*).

Answer: a) When $q = 0$, the second equation in system (*) becomes $q' = 0$. Hence q stays constantly equal to 0, the remaining two equations become

$$\begin{aligned} p' &= p(-1 + r) \\ r' &= r(1 - bp) \end{aligned}$$

Again this is a standard predator-prey system.

Answer: b) The equilibrium point is $(1/b, 1)$.

Answer: c) With $q = 0$ the Jacobian becomes

$$J = \begin{pmatrix} -1 + r & p(1 - r) & p \\ 0 & 1 - 2p & 0 \\ -br & brp & 1 - bp \end{pmatrix}$$

Then with $p = 1/b$ and $r = 1$, this becomes

$$J = \begin{pmatrix} 0 & 0 & 1/b \\ 0 & 1 - 2/b & 0 \\ -b & 1 & 0 \end{pmatrix}$$

It is not difficult to discover that the eigenvalues are $\pm i$ and $1 - 2/b$. If $b > 2$, e.g. if $b = 3$, then there is a positive eigenvalue and the equilibrium point is not a sink. If $b < 2$, then one eigenvalue is negative, but the other two have real part equal to 0. In this case it is impossible to decide.

Again it is highly likely that students used MATLAB to do these computations.

8. (11 points) For $b = 1$, use ode45 to compute the solution to system (*) for a variety of initial conditions over the time interval $[0, 50]$. Include the initial conditions $p(0) = q(0) = r(0) = 1$, and $p(0) = 1, q(0) = 1/2, r(0) = 1/3$. Submit only the plots of these two solutions. There should be one plot for each case with all three components plotted against time.

On the basis of your experimentation give a conjecture about the long term behavior of each of the three components of the general solution. Two sentences should suffice. In addition relate your answer to what you observed in Problems 6 & 7.

Answer: It is required to write a derivative m-file that describes the system. Here is an example of what that file can look like. (It is not required that the students submit such a file.)

```

function upr = pop(t,u)

global BB

p = u(1);
q = u(2);
r = u(3);

ppr = p*(-1 + q + exp(-q)*r);    % blue when plotted
qpr = q*(1 - 2*p);               % green when plotted
rpr = r*(1 - BB*exp(-q)*p);      % red when plotted

upr = [ppr;qpr;rpr];              % The semicolons make this a column
                                     % vector.

```

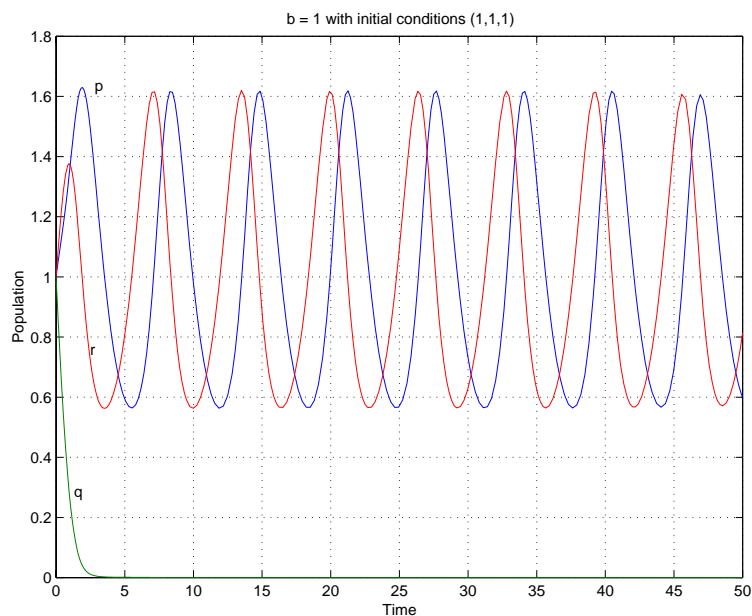


Figure 3.

The results of the computations for the two required cases are shown in Figures 3 & 4.

For these and other examples the prey population q dies out. The other two populations oscillate as a standard predator-prey system would. In fact they seem to oscillate around the point $p = 1$, $r = 1$, like the predator-prey system in Problem 7 with $b = 1$ does.

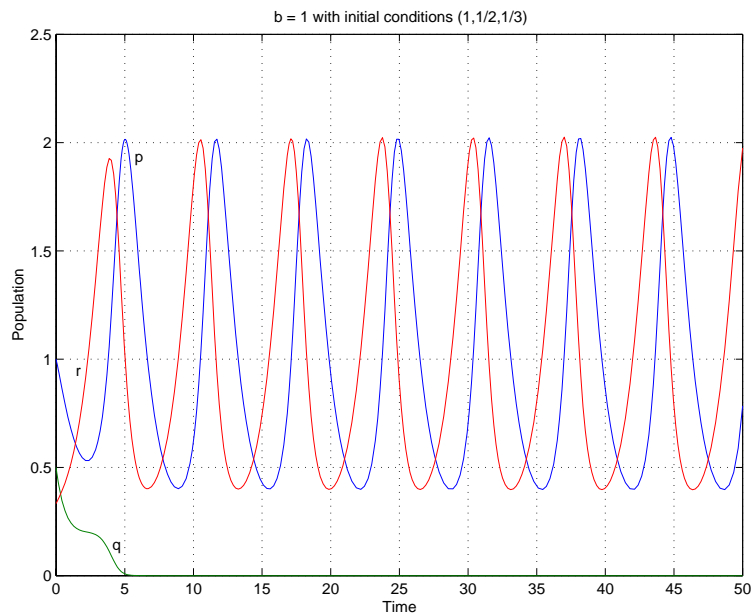


Figure 4.

9. (11 points) Repeat Problem 8 with $b = 3$.

Answer: We can use the same function m-file as before with $BB=3$. Our two plots are shown in Figures 5 & 6.

In this case we see that the prey population r seems to die out. It is not easy to make a prediction about the other populations. (It is not required to notice this, but looking at the solutions over a longer period shows that r dies out and p and q become periodic.) The equilibrium points in Problems 6 & 7 are both unstable for $b = 3$ so we would not expect them to come to bear here and indeed they do not.

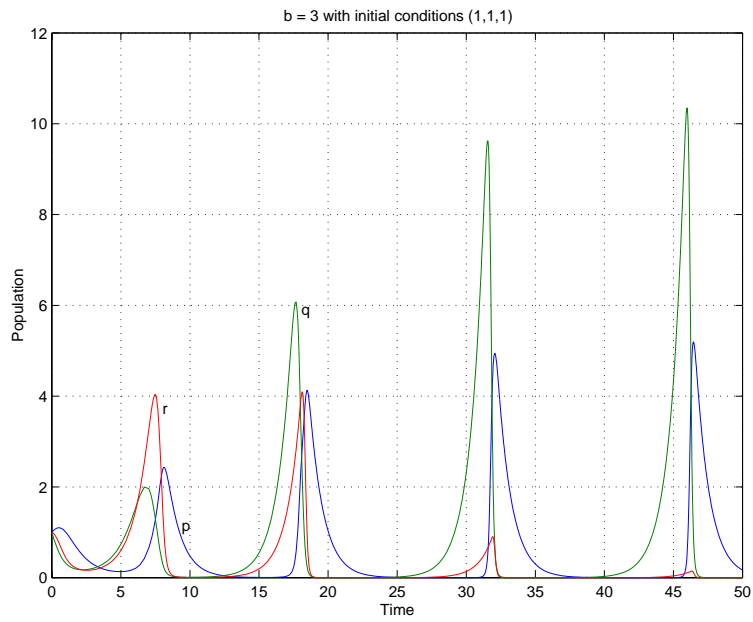


Figure 5.

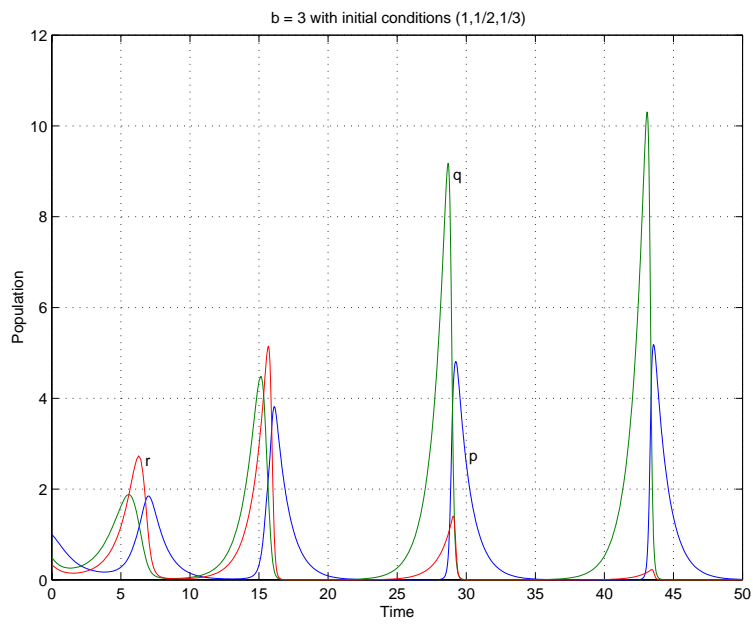


Figure 6.