Name:

Pledge:

Math 211

Final Exam

Fall 1996

Instructions: This is a closed book, 3 hour exam. Write out and sign the honor pledge on this page. You may use a calculator, but you may not use any symbolic computing capabilities of your calculator.

Circle the name of your instructor:

Polking Prokhorenkov Scannell Ward

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1. (7 points) Solve the given initial value problem

$$\frac{dy}{dt} = y^2t^3 + t^3,$$

$$y(0) = 1.$$

First, separate the variables and integrate:

$$\frac{dy}{y^2+1} = t^3 dt$$
$$arctan(y) = \frac{t^4}{4} + C$$
$$y(t) = tan(\frac{t^4}{4} + C).$$

Using the initial condition, we find $C = \pi/4$, so

$$y(t) = tan(\frac{t^4}{4} + \pi/4)$$

2. (8 points) Find the general solution of the following differential equation

$$\frac{dy}{dt} = -\frac{y}{t} + 2.$$

$$\begin{aligned}
\frac{dy}{dt} + \frac{y}{t} &= 2 \\
t\frac{dy}{dt} + y &= 2t \\
\frac{d}{dt}(yt) &= 2t \\
yt &= t^2 + C \\
y(t) &= t + C/t.
\end{aligned}$$

3. (10 points) Using Euler's method with the step size $\triangle t = 1$ compute an approximate solution to the initial value problem

$$\frac{dy}{dt} = y + t.$$

$$y(0) = 4.$$

over the interval $0 \leq t \leq 3$.

$$\begin{aligned} y(0) &\approx 4 \\ y(1) &\approx 4 + (4+0) = 8 \\ y(2) &\approx 8 + (8+1) = 17 \\ y(3) &\approx 17 + (17+2) = 36 \\ y(4) &\approx 36 + (36+3) = 75 \end{aligned}$$

4. (10 points) Consider the system of differential equations

$$\left\{ \begin{array}{l} \dfrac{dx}{dt} = y^2 - y \\ \dfrac{dy}{dt} = x^2 + 2x. \end{array} \right\}$$

Determine whether or not this system is Hamiltonian. If so, find a Hamiltonian function.

Let $f(x, y) = y^2 - y$, and $g(x, y) = x^2 + 2x$. Then this system is Hamiltonian because

$$\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y} = 0.$$

To find a Hamiltonian function $H(x, y)$:

$$\begin{aligned} \frac{\partial H}{\partial y} &= f(x, y) = y^2 - y \\ H(x, y) &= y^3/3 - y^2/2 + \phi(x), \end{aligned}$$

for some function $\phi(x)$ of x.

$$\begin{aligned} \frac{\partial H}{\partial x} &= -g(x, y) = -x^2 - 2x = \phi'(x) \\ \phi(x) &= -x^3/3 - x^2 \\ H(x, y) &= y^3/3 - y^2/2 - x^3/3 - x^2. \end{aligned}$$

5. Consider the system of differential equations

$$\left\{ \begin{array}{l} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = -2x - 3y. \end{array} \right\}$$

(a) (5 points) Show that all solutions of the system tend toward the origin as t increases to infinity, and sketch the phase plane.

This is a linear system with coefficient matrix:

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}.$$

We have:

$$A - \lambda I = \begin{pmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{pmatrix}.$$

The determinant of this matrix is $\lambda^2 + 3\lambda + 2$, and so the eigenvalues of A are -1 and -2. Since these are both negative, all solutions tend toward the sink at the origin.

(b) (5 points) Let

$$L(x, y) = x^2 + \frac{y^2}{2}.$$

Show that for any solution curve $(x(t), y(t))$,

$$\frac{d}{dt} L(x(t), y(t)) \leq 0.$$

Compute:

$$\begin{aligned} \frac{dL}{dt} &= \frac{\partial L}{\partial x}\frac{dx}{dt} + \frac{\partial L}{\partial y}\frac{dy}{dt} \\ &= 2x(y) + y(-2x - 3y) \\ &= -3y^2 \end{aligned}$$

which is always less than or equal to zero.

6. (7 points) There is a **non-zero** value of a such that the matrix equation

$$\begin{pmatrix} a & 1 & 4 \\ a & a & 6 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has a non-trivial solution. Compute this value of a, and find a basis for the nullspace in this case.

The determinant of the $3x3$ matrix can be computed easily, and is equal to $a^2 - 3a$. This determinant is zero when $a = 0$ or $a = 3$. We are only concerned with the non-zero solution.

When $a = 3$, the matrix becomes:

$$\begin{pmatrix} 3 & 1 & 4 \\ 3 & 3 & 6 \\ 0 & 1 & 1 \end{pmatrix}$$

which is row-reduced in the usual manner to:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The general solution to the given matrix equation is of the form $x = -z$, $y = -z$, and $z = z$. The nullspace is therefore spanned by the single vector

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

7. A damped pendulum is modeled by the system

$$\frac{dy}{dt} = v;$$
$$\frac{dv}{dt} = -\sin(y) - 2v.$$

a) (4 points) Find the equilibrium points.

The equilibria occur when $v = 0$, and $\sin(y) = 0$. This happens for $y = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$

b) (6 points) Use the technique of linearization to classify all equilibrium points with y-value between -5 and 5.

The Jacobian matrix for this system is given by:

$$\begin{pmatrix} 0 & 1 \\ -\cos(y) & -2 \end{pmatrix}$$

The only equilibrium points of interest are $(-\pi, 0), (0, 0)$, and $(\pi, 0)$. The linear approximation to the system at $(0, 0)$ is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$$

which has negative eigenvalues, implying a sink.

The linear approximations at the other two points are the same, given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}.$$

The eigenvalues in this case are real and of opposite sign, implying saddle points at $(-\pi, 0)$ and $(\pi, 0)$.

8. Consider the linear system

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

a) (6 points) Find the general solution of this system.

Let $A = \begin{pmatrix} 2 & 1 \\ -2 & 5 \end{pmatrix}$. Then $det(A - \lambda I) = \lambda^2 - 7\lambda + 12 = (\lambda - 4)(\lambda - 3)$.

Any non-zero vector in the nullspace of $A - 3I = \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}$ is an eigenvector for the eigenvalue $\lambda = 3$. In particular, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ works.

Similarly, any non-zero vector in the nullspace of $A - 4I = \begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix}$ is an eigenvector for the eigenvalue $\lambda = 4$. We choose $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Thus the general solution to the linear system is given by

$$\mathbf{Y}(t) = k_1 e^{4t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + k_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

b) (3 points) Find the solution with initial value $x(0) = 1$, $y(0) = 0$.

We require:

$$\mathbf{Y}(0) = k_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Taking $k_1 = -1$ and $k_2 = 2$ gives the desired particular solution.

c) (3 points) Sketch the phase plane.

9. (8 points) Find the general solution of the 2nd order equation

$$\frac{d^2y}{dt^2} + y = 20e^{3t}.$$

We begin by solving the homogeneous equation

$$\frac{d^2y}{dt^2} + y = 0.$$

This is completely routine, in fact, it is clear by inspection that $\sin(t)$ and $\cos(t)$ are linearly independent solutions. Thus

$$y_h(t) = k_1 \cos(t) + k_2 \sin(t)$$

is the general solution to the homogeneous equation.

Next, we need a particular solution to the inhomogeneous equation. For this, we guess a solution of the form ae^{3t}. Plugging in, we get

$$\begin{aligned} 9ae^{3t} + ae^{3t} &= 20e^{3t} \\ 10a &= 20 \\ a &= 2. \end{aligned}$$

Thus a particular solution is $2e^{3t}$, and the general solution to the inhomogeneous equation is

$$y(t) = k_1 \cos(t) + k_2 \sin(t) + 2e^{3t}.$$

10. Consider the system

$$\frac{dx}{dt} = x\,(20 - x - y);$$
$$\frac{dy}{dt} = y\,(30 - 2x - y).$$

a) (5 points) Sketch the phase plane for this system, showing the nullclines, the equilibrium points, and some typical solutions.

The sketch is almost identical to Figure 2.67 on page 202 of BDH, scaling the axes appropriately. The equilibria are at $(0,0)$, $(0,30)$, $(20,0)$, and $(10,10)$.

b) (5 points) Classify the equilibrium points. Justify your answers.

The point $(0,0)$ is a source, $(10,10)$ is a saddle, and the other two are sinks. See the discussion on p. 202 of BDH.

c) (3 points) Give two different points in the phase plane so that the solutions which pass through these points have different long-term behaviors.

Starting at the initial condition $(0,100)$, the solution will head toward the equilibrium point at $(0,30)$, while if we start at $(100,0)$, the soultion tends toward $(20,0)$.

d) (3 points) Would this system better describe a predator-prey or competing species population model? Why?

This system is a better model of a competing-species situation. A large increase in one species forces the population of the other species to decline; this is unlike an increase in the number of prey, which we would expect to be beneficial to the predators. See section 2.6 of BDH for more information.

11. (7 points) The unforced vibration of the dashboard in a certain math professor's station wagon is modeled by:

$$\frac{d^2x}{dt^2} + 9x = 0,$$

where the damping term is considered negligible.

When the car is being driven at K miles per hour, there is a periodic forcing on the dashboard, given by the function

$$f(t) = 0.1 \sin\left(\frac{K}{20}t\right).$$

What speed K will cause the dashboard to vibrate in resonance?

Without even solving the homogeneous system, we can simply compute the eigenvalues of the associated linear system, obtaining complex eignevalues $3i$ and $-3i$. Thus the solution will involve terms like $\sin(3t)$ and $\cos(3t)$, a natural frequency of $\omega = 3$. Resonance will therefore occur when this natural frequency matches the forcing frequency, that is, when $\frac{K}{20} = 3$, or $K = 60$ miles per hour.

12. A singer is trying to shatter a wine glass; the vibration is modeled by the forced harmonic oscillator equation:

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 6\cos(\sqrt{2}\cdot t).$$

where x is the displacement near the top of the glass (measured in millimeters).

If $|x(t)|$ ever exceeds 6, the glass shatters.

(a) (7 points) The general solution to the **homogeneous** equation is

$$c_1e^{-t}\cos(t) + c_2e^{-t}\sin(t).$$

What is the solution to the **nonhomogeneous** equation with initial conditions $x(0) = 0$, $x'(0) = 0$?

We need to find a particular solution. We do this by first finding a particular solution to the complex equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 6e^{i\sqrt{2}t}.$$

A good guess is a solution of the form $ae^{i\sqrt{2}t}$. Plugging in, we get:

$$\begin{aligned} -2ae^{i\sqrt{2}t} + 2\sqrt{2}iae^{i\sqrt{2}t} + 2ae^{i\sqrt{2}t} &= 6e^{i\sqrt{2}t} \\ -2a + 2\sqrt{2}ia + 2a &= 6 \\ a(2\sqrt{2}i) &= 6 \\ a &= \frac{-3\sqrt{2}}{2}i. \end{aligned}$$

Taking the real part of the particular complex solution, we obtain $x_p(t) = \frac{3\sqrt{2}}{2}\sin(\sqrt{2}t)$. The general solution is:

$$x(t) = c_1e^{-t}\cos(t) + c_2e^{-t}\sin(t) + \frac{3\sqrt{2}}{2}\sin(\sqrt{2}t).$$

When $t = 0$, we get $c_1 = 0$ from the initial condition. Taking the derivative,

$$x'(t) = c_2e^{-t}(\cos(t) - \sin(t)) + 3\cos(\sqrt{2}t),$$

and using the other initial condition, we get $c_2 + 3 = 0$, so $c_2 = -3$. Our answer is therefore

$$x(t) = 3e^{-t}\sin(t) + \frac{3\sqrt{2}}{2}\sin(\sqrt{2}t).$$

(b) (4 points) Describe the long term behavior of the solution with these initial conditions.

This solution tends toward the particular solution, since e^{-t} goes to zero as t goes to infinity (this is the damping effect).

(c) (4 points) Will the glass shatter?

No. The absolute value of $x(t)$ is bounded above by $3 + \frac{3\sqrt{2}}{2}$ which is less than 6.