Math 211

Final Exam

December 9, 1995

Part 2

Instructions: Write out and sign the honor pledge on your exam paper for Part 2. In addition put the name of your instructor in a prominent place on your exam. It is due by 3:30 PM on Friday, December 20, in the Mathematics Department Office, HB 220.

Part 1 is worth 120 points, and Part 2 is worth 80 points.

Part 2 of the exam is an open book, open notes, take home exam. There is no time limit. While the exam emphasizes the computer aspect of the course, you are allowed to use any of the analytic methods discussed in the course. In fact you will be required to.

The first part of the exam examines a predator prey situation. We will denote the prey population by $x(t)$ and the predator population by $y(t)$. The new feature of this model is that we will separate the prey into young prey with population $x_1(t)$, and adult prey with population $x_2(t)$. Thus $x(t) = x_1(t) + x_2(t)$.

We will assume, for whatever reason, that the predators only prey on the adults. Thus the young prey increase because of births, which are proportional to the number of adults, and they decrease because of deaths, which is proportional to the number of young, and because they become adults. The last change is again proportional to the number of young. Hence there are positive constants a, a_1, and a_2 such that

$$x_1' = ax_2 - a_1x_1 - a_2x_1.$$

The adult prey increase because the youth mature (the term a_1x_1 in the above equation). They decrease because they are preyed upon and because they die of natural causes. Hence there are positive constants a_3, and b such that

$$x_2' = a_1x_1 - a_3x_2 - bx_2y.$$

Finally the change in the predator population is the same as usual. There are positive constants c and d such that

$$y' = -cy + dx_2y.$$

We will consider the case when $a = 2$, $a_1 = a_2 = a_3 = 1/2$, and $b = c = d = 1$. Hence the three equations become

$$\begin{aligned} x_1' &= 2x_2 - \frac{1}{2}x_1 - \frac{1}{2}x_1 \\ x_2' &= \frac{1}{2}x_1 - \frac{1}{2}x_2 - x_2y \\ y' &= -y + x_2y. \end{aligned}$$

1. (10 points) First examine what happens in the absence of the predators (i.e., when $y = 0$). The resulting 2-dimensional system can be examined analytically or by using `pplane`. Show that for meaningful initial conditions, the total prey population $x(t)$ tends to ∞ as t tends to ∞.

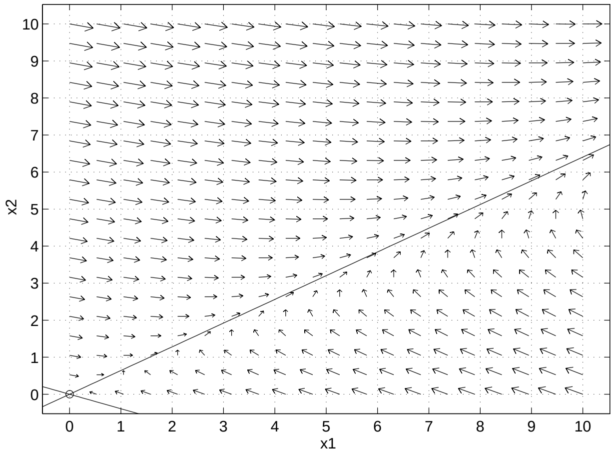

Figure 1. The separatrices of the linear system.

Answer: With $y = 0$ we have the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where

$$A = \begin{pmatrix} -1 & 2 \\ 1/2 & -1/2 \end{pmatrix}.$$

A has eigenvalues $(-1 \pm \sqrt(3))/2$. In MATLAB we get

```
>> [V,E] = eig(A)

V =

   -0.9315   -0.8421
    0.3637   -0.5393


E =

   -1.7808         0
         0    0.2808
```

Since the eigenvalues are real and have different signs, the origin is a saddle point. An eigenvector for the positive eigenvalue is $\begin{pmatrix} 0.8421 \\ 0.5393 \end{pmatrix}$, which points into the first quadrant. The eigenvectors for the negative eigenvalue point into the second and fourth quadrants. Since solutions go to infinity along the eigenvectors for the positive eigenvalue, we see that all solutions which start in the first quadrant must go to infinity.

This is illustrated in Figure 1, which plots the stright line solutions (separatrices) for the system. There are a number of ways to come to this answer.

2. (10 points) Find the equilibrium points for the full 3-dimensional system. There are two.

Answer: The equilibrium points must satisfy the three equations

$$\begin{aligned} 2x_2 - x_1 &= 0 \\ x_1/2 - x_2/2 - x_2 y &= 0 \, . \\ -y + x_2 y &= 0 \end{aligned}$$

The third equation implies that $y = 0$ or $x_2 = 1$. Substituting these two values into the other equations we easily see that there are two solutions: $x_1 = x_2 = y = 0$, and $x_1 = 2$, $x_2 = 1$, $y = 1/2$.

3. (10 points) Find the Jacobian matrix for the full 3-dimensional system at each of the equilibrium points, and use this information to classify the equilibrium point. We want to know what happens as t tends to ∞ to solutions which start near these points.

Answer: By differentiating we find that the jacobian is

$$J = \begin{pmatrix} -1 & 2 & 0 \\ 1/2 & -1/2 - y & -x_2 \\ 0 & y & x_2 - 1 \end{pmatrix} .$$

At the first equilibrium point we have

$$J(0,0,0) = \begin{pmatrix} -1 & 2 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & -1 \end{pmatrix} .$$

Using MATLAB we find

```
>> J=[-1 2 0;1/2 -1/2 0;0 0 -1]

J =

   -1.0000    2.0000         0
    0.5000   -0.5000         0
         0         0   -1.0000
```

```
>> eig(J)

ans =

   -1.7808
    0.2808
   -1.0000
```

Thus the jacobian has one positive and two negative eigenvalues. Since at least one of the eigenvalues is positive, the equilibrium point at the origin is unstable, and most solutions which start near the origin do not converge to the origin as $t \to \infty$.

At the second equilibrium point we have

$$J(2,1,1/2) = \begin{pmatrix} -1 & 2 & 0 \\ 1/2 & -1 & -1 \\ 0 & 1/2 & 0 \end{pmatrix}.$$

Using MATLAB

```
>> J = [-1 2 0;1/2 -1 -1;0 1/2 0]

J =

   -1.0000    2.0000         0
    0.5000   -1.0000   -1.0000
         0    0.5000         0

>> eig(J)

ans =

  -1.8756
  -0.0622 + 0.5126i
  -0.0622 - 0.5126i
```

Thus all of the eigenvalues of the jacobian have negative real part. Consequently this equilibrium point is a sink, and all orbits that start near to it converge to the equilibrium point as $t \to \infty$.

4. (10 points) Use `ode45` to solve the full 3-dimensional system for the 2 sets of initial conditions, $x_1(0) = 0.2$, $x_2(0) = 0.1$, $y(0) = 1$, and $x_1(0) = 2$, $x_2(0) = 3$, $y(0) = 1$. Turn in two plots, one for each solution. What happens as t tends to ∞? Your time interval should be long enough for you to easily decide the answer to this question.

Answer: The plots In Figures 2 and 3 clearly show that in both cases the solutins converge to the equilibrium point as $t \to \infty$.

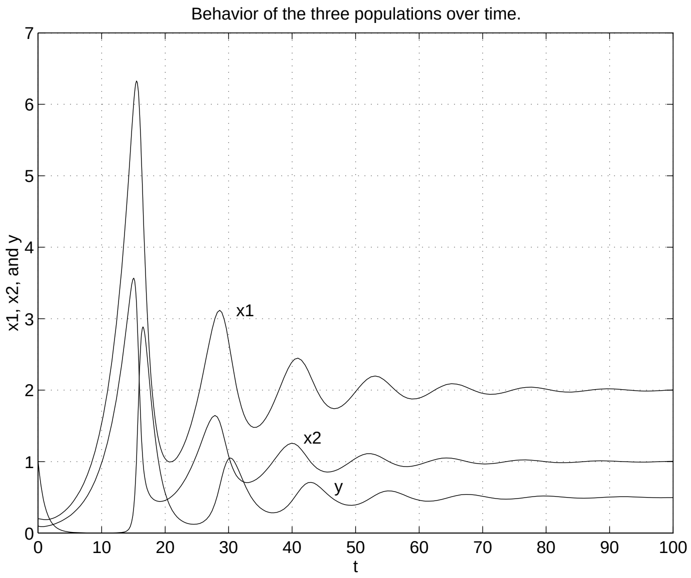

Figure 2. The populations starting at $x_1 = 0.2$, $x_2 = 0.1$, and $y = 1$.

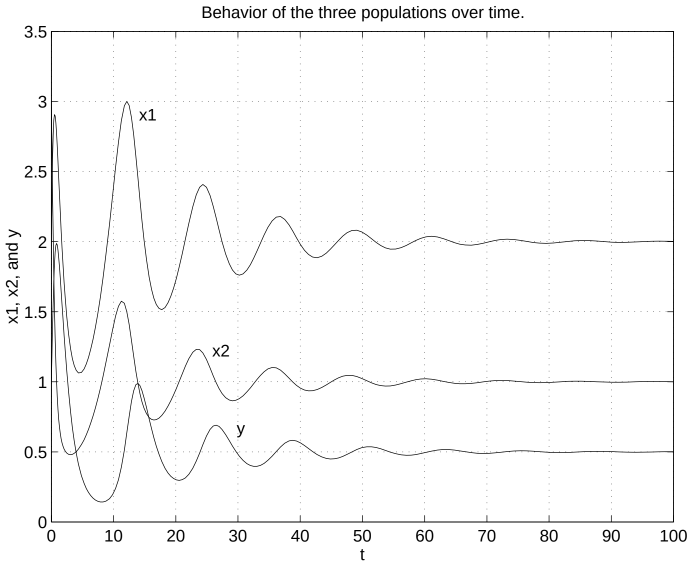

Figure 3. The populations starting at $x_1 = 2$, $x_2 = 3$, and $y = 1$.

5. (10 points) For the first solution in Problem 4, plot y versus x, where $x = x_1 + x_2$. Remember that MATLAB can easily add vectors.

Answer: This can be easily accomplished with the follwing MATLAB comands.

```
>> [t,u] = ode45('f964',0,100,[0.2 0.1 1]);
>> plot(u(:,1)+u(:,2),u(:,3))
```

The result is shown in Figure 4.

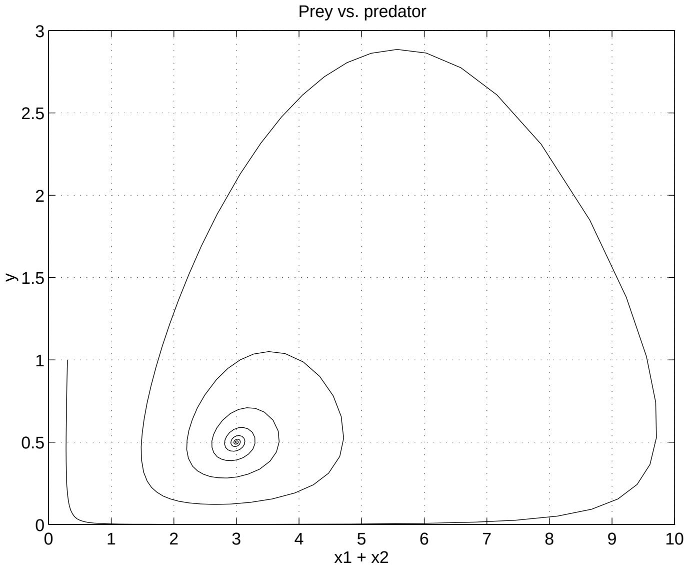

Figure 4. Predator population plotted against total prey population.

The remainder of the test will be devoted to the study of the forced pendulum. You will remember that in class we discovered a lot of facts about the forced oscillator. That phenomenon is linear, and therefore could be analyzed in some detail. The pendulum, however, is nonlinear, and for that reason is much harder to analyze. We are almost forced to use numerical methods. As you go through this exercise you should compare the results you find with what happens in the linear case.

We will consider the situation where the pivot of the pendulum is forced to move, and the position is given by its x and y coordinates denoted by the captial letters $X(t)$ and $Y(t)$. We will not go into detail about the modeling here. The equation that describes the motion of the pendulum bob is

$$L\theta'' + X''(t)\cos(\theta) + Y''(t)\sin(\theta) + g\sin(\theta) = 0,$$

where θ is the angle of the pendulum with the vertical, L is the length of the pendulum, and g is acceleration due to gravity. The term $X''(t)$ is the second derivative of X.

You will notice that there is no damping term, so we are looking at undamped motion of the forced pendulum.

There are a wide variety of motions that we could impose upon the pivot, but we will limit ourselves to the case when $X(t) = 0.1\cos(\omega t)$, and $Y(t) = 0$. Thus we are oscillating the pivot back and forth in the horizontal direction with frequency ω, but with a relatively small amplitude. To make things simpler we will assume that we are using coordinates in which $L = 1$ and $g = 1$. Hence the equation we will be solving is

$$\theta'' - 0.1\,\omega^2 \cos(\omega t)\cos(\theta) + \sin(\theta) = 0.$$

It is a good idea to imagine that you are holding the pivot in your hand, and to try to think out what will happen. You can then compare your mental picture with the results of the following computational experiment. You may find the results counterintuitive, however.

6. (15 points) Solve the equation with initial conditions $\theta(0) = \theta'(0) = 0$, and with $\omega = 0.5$, 8 and 20. In each case plot θ versus t (information about θ' is not of much interest). Examine different ranges of t. Remember that shorter ranges than computed can be examined using the `axis` command. Describe the phenomenon you see and compare it to the linear case. You will undoubtedly look at lots of graphs, but turn in only the one plot of θ versus t which you find most interesting. Be sure to label the plot so that the grader will know the case it refers to.

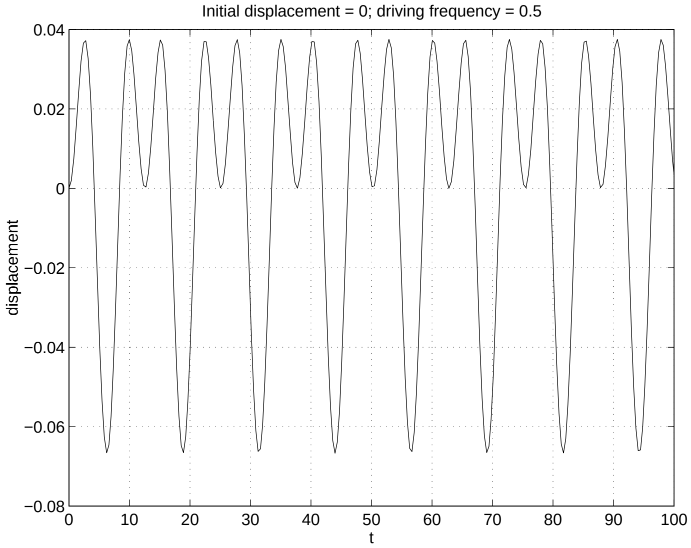

Figure 5. Driving frequency $\omega = 0.5$.

Answer: In the case of the forced linear oscillator without damping we get periodic oscillations. The oscillations are a mixture of the natural frequency and forcing frequency. Sometimes this is pretty complicated leading to the phenomenon of beats. In all cases the oscillations are symmetric about 0 displacement. In the case of the forced pendulum we again see periodic behavior with a mixture of the natural frequency and forcing frequency. Again we see something that looks like beats. Now, however, the oscillations at the low and at the high frequencies are skewed to the positive, while at $\omega = 8$ we get symmetric oscillations. The most remarkable case is at $\omega = 20$, where the pendulum bob seems never to pass to a negative angle.

To allow for the choices made by the students we will provide plots for all three cases.

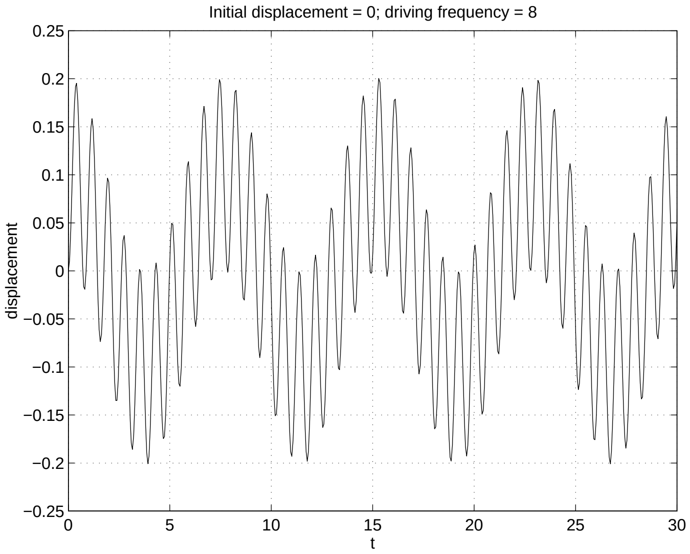

Figure 6. Driving frequency $\omega = 8$.

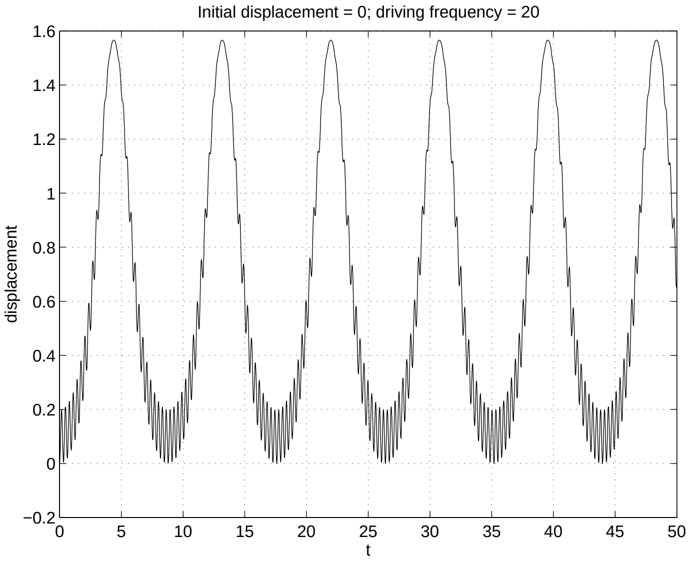

Figure 7. Driving frequency $\omega = 20$.

7. (15 points) Solve the equation with initial conditions $\theta(0) = 1$, and $\theta'(0) = 0$, and with $\omega = 0.5$, 8 and 20. In each case plot θ versus t (information about θ' is not of much interest). Examine different ranges of t. Remember that shorter ranges than computed can be examined using the `axis` command. Describe the phenomenon you see and compare it to the linear case, and to the case in the previous problem. You will undoubtedly look at lots of graphs, but turn in only the one plot of θ versus t which you find most interesting. Be sure to label the plot so that the grader will know the case it refers to.

Answer: Even with a nonzero initial displacement, in the linear case we expect an oscillation similar in many ways to that with zero initial displacement. By the linearity we know that the displacement will be a linear combination of an osciallation at the natural frequency and one at the driving frequency. Even with a large initial displacement the oscillation will be symmetric with respect to the origin.

For the two low frequencies this is approximately what we see. The result for $\omega = 20$ is quite different. In this case we see an oscillation at the driving frequency, but this is modulated by the natural frequency. The most surprising feature is that the oscillation leaves the pendulum bob at an angle to the perpendicular of close to 1 radian at all times. Nothing about linear theory would lead us to expect that. The very rapid oscillation of the pivot is completely overwhelming the force of gravity.

Again we will provide graphs of all three cases.

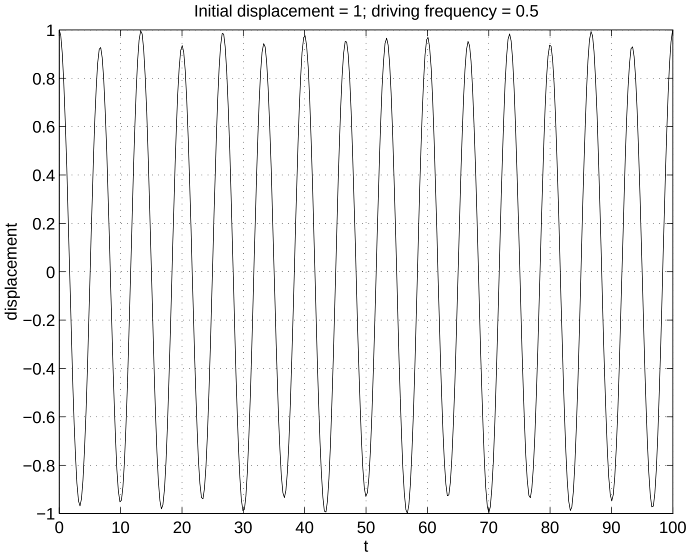

Figure 8. Driving frequency $\omega = 0.5$.

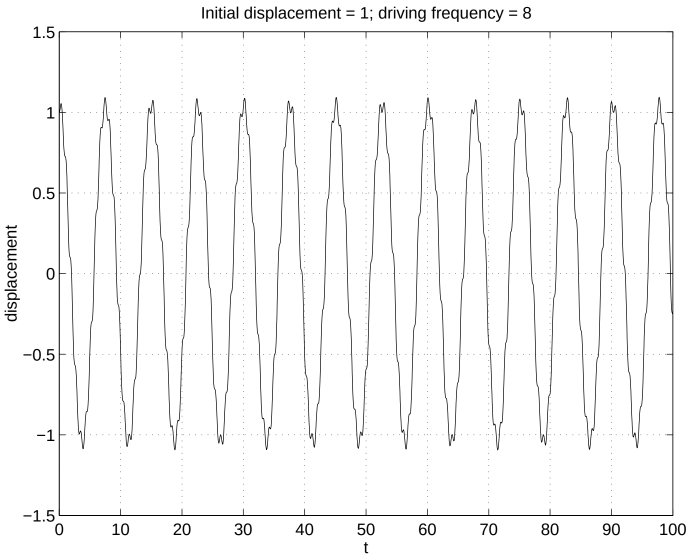

Figure 9. Driving frequency $\omega = 8$.

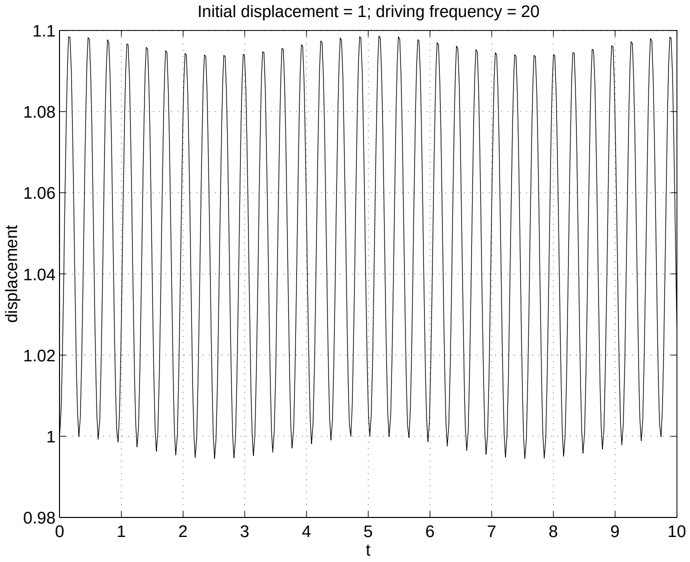

Figure 10. Driving frequency $\omega = 20$.