

Math 211

Exam # 2

November 3, 1996

Part 2

Instructions: Part 2 of Exam #1 is an open book, open notes, untimed, take home exam. It is due in the Mathematics Department Office by 3:30 PM on Friday, November 8. You may not consult with your fellow students about the exam. If you have any questions, consult with one of the instructors for the course.

Write out and sign the pledge on your submitted solutions. Be sure to put your name on your submission. In addition put the name of your instructor in a prominent place on the first page of your submission.

Please give reasons for all of your answers. Remember that some reasons are better than others. For example, it is better to refer to a theorem stated in class and/or in the books than to say “the computer printout shows that ...”. Of course sometimes the computer printout is all you have.

In answering these questions you are not limited to the use of MATLAB, although it will be useful. You should use any combination of the analytic, qualitative, or numerical methods you have learned.

Probably most of you will want to use a MATLAB diary file in answering these questions. The ideal way to prepare your submission is to use an editor to insert your comments into the diary file. If you use the editor to leave enough room, equations can be entered by hand. If you do not want to use an editor it is still your responsibility to organize your work in a readable and orderly manner. Your comments should consist of complete sentences organized into cohesive paragraphs. This does not mean that they have to be lengthy. In fact brevity is frequently a sign of understanding. If you include MATLAB graphics they should be numbered and referred to by that number. Any graphic which is not referred to will not be counted as part of your submission.

The matrices **A**, **B**, and **C** and vectors \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 , \mathbf{v}_4 , and \mathbf{v}_5 can be loaded into your MATLAB workspace with the command `load exam2data`.

For the first two problems we will consider the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & -2 & -1 & 0 & 0 \\ 11 & -1 & 4 & 1 & -2 \\ 3 & 1 & 14 & 2 & 0 \end{pmatrix}.$$

1. Find a basis for the nullspace of **A**.

Answer: From the diary file:

```
>> rref(A)
```

```
ans =
```

```
1.0000    0    4.5000    0    4.0000
         0    1.0000    0.5000    0         0
         0         0         0    1.0000   -6.0000
```

We see from this that x_3 and x_5 are free variables. Setting $x_3 = 1$ and $x_5 = 0$, and solving for the rest we get $x_1 = -9/2$, $x_2 = -1/2$, and $x_4 = 0$. Next setting $x_3 = 0$ and $x_5 = 1$, and solving for the rest we get $x_1 = -4$, $x_2 = 0$, and $x_4 = 6$. Hence a basis is

$$\mathbf{v}_1 = \begin{pmatrix} -9/2 \\ -1/2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} -4 \\ 0 \\ 0 \\ 6 \\ 1 \end{pmatrix}.$$

This problem can also be solved using `null`, in which case the basis found will be more complicated, but still correct.

2. Find the general solution to the equation $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{b} = \begin{pmatrix} 10 \\ 7 \\ -7 \end{pmatrix}.$$

Answer: We form the augmented matrix

```
>> a = [A b]
```

```
a =
```

```
0    -2    -1    0    0    10
1    -1     4    1   -2     7
3     1    14    2    0    -7
```

To find the solutions we use `rref`

```
>> rref(a)
```

```
ans =
```

```
1.0000    0    4.5000    0    4.0000   -6.0000
         0    1.0000    0.5000    0         0   -5.0000
         0         0         0    1.0000   -6.0000    8.0000
```

We can set $x_3 = s$ and $x_5 = t$, and solve for the rest of the variables, or we can set $x_3 = x_5 = 0$ and find the particular solution

$$\mathbf{v}_0 = \begin{pmatrix} -6 \\ -5 \\ 0 \\ 8 \\ 0 \end{pmatrix}.$$

Either way we find that the general solution is

$$\mathbf{v}_0 + s\mathbf{v}_1 + t\mathbf{v}_2 \quad \text{where } s \text{ and } t \text{ are arbitrary.}$$

For the next three problems we will use the following vectors.

$$\mathbf{v}_1 = \begin{pmatrix} 4 \\ -1 \\ 2 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} -5 \\ 1 \\ -4 \\ -3 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_5 = \begin{pmatrix} 4 \\ 0 \\ 2 \\ 2 \end{pmatrix}.$$

3. Are the vectors \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 , and \mathbf{v}_4 linearly dependent or independent? If they are linearly dependent display a non-trivial linear combination which is equal to the zero vector. If they are linearly independent give a proof.

Answer: We form the matrix with the vectors \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 , and \mathbf{v}_4 as columns.

```
>> V=[v1 v2 v3 v4]
```

V =

$$\begin{array}{cccc} 4 & 2 & 1 & -5 \\ -1 & 1 & 1 & 1 \\ 2 & 2 & 0 & -4 \\ 2 & 2 & 1 & -3 \end{array}$$

With this matrix \mathbf{V} we know that a linear combination

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4 = \mathbf{V} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix}$$

Hence we can find a linear combination equal to the zero vector if and only if there is a nonzero

vector in the nullspace of V . To find the nullspace we use `rref`

```
>> rref(V)
```

```
ans =
```

```
1    0    0   -1
0    1    0   -1
0    0    1    1
0    0    0    0
```

We see that

$$\mathbf{k} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

is in the null space of V . Hence $\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$, and the vectors are linearly dependent.

4. Are the vectors $\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$, and \mathbf{v}_5 linearly dependent or independent? If they are linearly dependent display a non-trivial linear combination which is equal to the zero vector. If they are linearly independent give a proof.

Answer: We proceed as in the previous problem, but without comment.

```
>> V=[v2 v3 v4 v5]
```

```
V =
```

```
2    1   -5    4
1    1    1    0
2    0   -4    2
2    1   -3    2
```

```
>> rref(V)
```

```
ans =
```

```
1    0    0    0
0    1    0    0
0    0    1    0
0    0    0    1
```

Since the nullspace of V contains only the zero vector, the vectors are linearly independent.

Computing that the determinant of V is nonzero would also suffice.

5. Consider the system

$$\begin{aligned}x' &= y, \\y' &= x - x^3.\end{aligned}$$

This system is a special case of Duffing's equation. It models the motion of a pendulum with an iron bob when there are two strong magnets located along the arc of the pendulum; x is the angular displacement and y is the angular velocity.

a) On a piece of paper (preferably graph paper) draw a picture displaying the following items.

i) Draw the nullclines and indicate the direction of the vector field along the nullclines.

ii) Find and indicate the location of all of the equilibrium points.

It is probably easier to do this problem by hand, but if you can do it on the computer, go ahead.

Answer: The required information is shown on Figure 1. The x -nullcline is dashed and the y -nullcline is solid. The equilibrium points are shown as small circles, and the arrows indicate the direction of the vector field at the indicated points. Since the direction is constant along nullclines between equilibrium points, these few vectors suffice to show the entire picture.

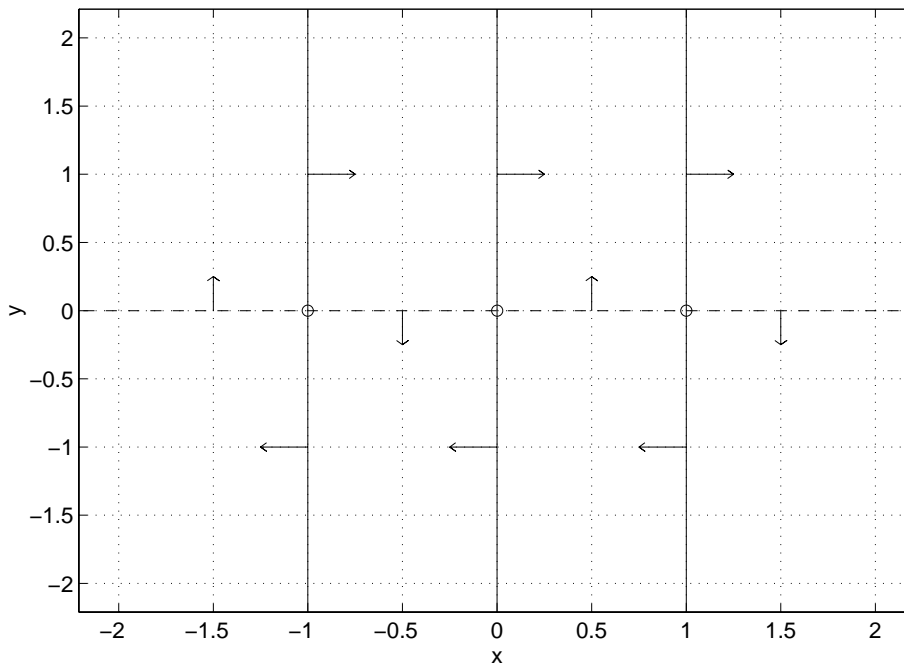


Figure 1 Nullclines and equilibrium points.

- b) On the basis of the information on this figure what can you say qualitatively about the solutions to the system?

Answer: We can see that the solution curves “circle” in a clockwise direction. Those that start near the equilibrium points at $(-1, 0)$ and $(1, 0)$ circle around those points, passing between the points and the origin. Those that start further away from these points seem to circle around all three equilibrium points. It is not possible to decide if the solution curves are closed (i.e., the solutions are periodic).

- c) Use PPLANE to compute and plot enough solutions to get a general idea about the behavior of all solutions. Print and submit the result. What additional information do you get about the solutions from this figure?

Answer:

With the information we have already obtained it is only necessary to plot the three solutions indicated in Figure 2. this verifies what we found in part b), and in addition shows us that the solution curves are closed.

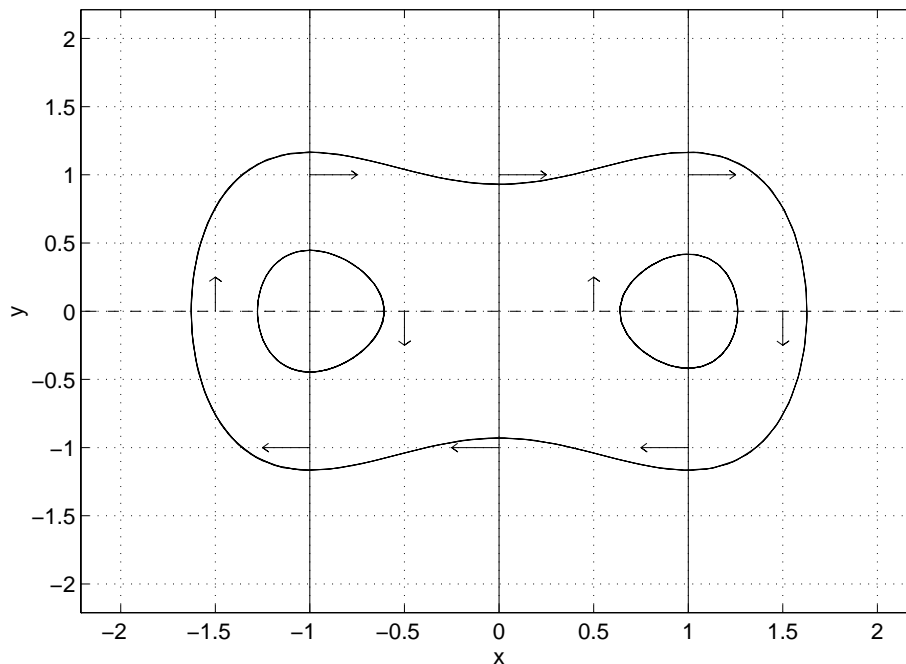


Figure 2 Three solution curves.

There has to be some transition between these types of solutions near the origin, but it is not necessary to describe that. For these answers it was convenient to plot the solution curves on

the same plot as was submitted for part a). This will not be possible for most of the students. A standard PPLANE plot will suffice.

6. Find the general solution to the system $\mathbf{x}' = \mathbf{B}\mathbf{x}$, where

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & -4 \\ 6 & -3 & 6 \\ 2 & 0 & 7 \end{pmatrix}.$$

Find the solution with the initial value

$$\mathbf{x}(0) = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$$

Answer: There are many ways to do these problems, and there are many different ways to express the answer. Here is one way. We first find the eigenvalues.

```
>> eig(B)
```

```
ans =
```

```
-3  
3  
5
```

Since the eigenvalues are real and distinct we know that for each eigenvalue λ with associated eigenvector \mathbf{v} we have the solution $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$. For the eigenvalue -3 we use `rref` to find an eigenvector.

```
>> rref(B-(-3)*eye(3))
```

```
ans =
```

```
1    0    0  
0    0    1  
0    0    0
```

We see that

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_1(t) = e^{-3t}\mathbf{v}_1 = \begin{pmatrix} 0 \\ e^{-3t} \\ 0 \end{pmatrix}.$$

Similarly for the eigenvalue 3:

```
>> rref(B-(3)*eye(3))
```

ans =

$$\begin{array}{ccc} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}$$

so

$$\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{3t} \mathbf{v}_2 = \begin{pmatrix} 2e^{3t} \\ e^{3t} \\ -e^{3t} \end{pmatrix}$$

Finally for the eigenvalue 5

```
>> rref(B-(5)*eye(3))
```

ans =

$$\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}$$

so

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_3(t) = e^{5t} \mathbf{v}_3 = \begin{pmatrix} e^{5t} \\ 0 \\ -e^{5t} \end{pmatrix}$$

The general solution is $\mathbf{x}(t) = k_1 \mathbf{x}_1(t) + k_2 \mathbf{x}_2(t) + k_3 \mathbf{x}_3(t)$.

To find the solution with the indicated initial value we must find the coefficients such that

$$\begin{aligned} \mathbf{x}(0) &= k_1 \mathbf{x}_1(0) + k_2 \mathbf{x}_2(0) + k_3 \mathbf{x}_3(0) \\ &= k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 + k_3 \mathbf{v}_3 \\ &= [\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3] \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} \end{aligned}$$

In MATLAB

```
>> v1=[0 1 0]';v2=[2 1 -1]';v3=[1 0 -1]';  
>> V=[v1 v2 v3]
```

V =

```
    0    2    1  
    1    1    0  
    0   -1   -1
```

```
>> x0=[4 0 -3]';  
>> k=V\x0
```

k =

```
   -1  
    1  
    2
```

Hence the solution is $\mathbf{x}(t) = -\mathbf{x}_1(t) + \mathbf{x}_2(t) + 2\mathbf{x}_3(t)$.

It is also possible to do this problem using $[V, E] = \text{eig}(B)$. If the problem is done this way, the eigenvectors may be multiples of the ones found here, and as the result the coefficients in the vector \mathbf{k} will be different.

7. Consider the matrix

$$\mathbf{C} = \begin{pmatrix} -3 & -4 & 2 & 0 \\ -5 & -6 & 5 & 1 \\ -9 & -11 & 8 & 1 \\ -4 & -6 & 4 & 1 \end{pmatrix}.$$

a) Find the general solution to the system $\mathbf{x}' = \mathbf{C}\mathbf{x}$.

Answer: We use `eig` to find the eigenvalues and eigenvectors.

```
>> [V,E] = eig(C)
```

V =

```
    0.7071         0.3162         0.3634 - 0.4486i    0.3634 + 0.4486i  
   -0.0000        -0.6325        -0.4060 + 0.0426i    -0.4060 - 0.0426i  
    0.7071        -0.6325        -0.0426 - 0.4060i    -0.0426 + 0.4060i  
   -0.0000         0.3162        -0.4486 - 0.3634i    -0.4486 + 0.3634i
```

E =

$$\begin{array}{cccc} -1.0000 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0.0000 + 1.0000i & 0 \\ 0 & 0 & 0 & 0.0000 - 1.0000i \end{array}$$

Just to be fancy we will look for eigenvectors with integer entries. For the eigenvalue -1 we have

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_1(t) = e^{-t} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

and for the eigenvalue +1 we have

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^t \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix}$$

For the complex eigenvalue i we use MATLAB

```
>> w = V(:,3)/V(3,3)
```

w =

$$\begin{array}{l} 1.0000 + 1.0000i \\ 0.0000 - 1.0000i \\ 1.0000 \\ 1.0000 - 1.0000i \end{array}$$

which yields the complex solution

$$\begin{aligned} \mathbf{z}(t) &= e^{it} \begin{pmatrix} 1+i \\ -i \\ 1 \\ 1-i \end{pmatrix} \\ &= [\cos t + i \sin t] \left[\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix} \right] \\ &= \begin{pmatrix} \cos t - \sin t \\ \sin t \\ \cos t \\ \cos t + \sin t \end{pmatrix} + i \begin{pmatrix} \cos t + \sin t \\ -\cos t \\ \sin t \\ \cos t + \sin t \end{pmatrix} \end{aligned}$$

Since the real and imaginary parts of a complex valued solution are solutions we have the real valued solutions

$$\mathbf{x}_3(t) = \begin{pmatrix} \cos t - \sin t \\ \sin t \\ \cos t \\ \cos t + \sin t \end{pmatrix} \quad \text{and} \quad \mathbf{x}_4(t) = \begin{pmatrix} \cos t + \sin t \\ -\cos t \\ \sin t \\ -\cos t + \sin t \end{pmatrix}.$$

The general solution is $\mathbf{x}(t) = k_1\mathbf{x}_1(t) + k_2\mathbf{x}_2(t) + k_3\mathbf{x}_3(t) + k_4\mathbf{x}_4(t)$.

Once again there are different but equally valid ways of doing the problem.

b) Find the solution to the system $\mathbf{x}' = \mathbf{C}\mathbf{x}$, with the initial value

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}.$$

Answer: We must find the coefficients such that $\mathbf{x}(0) = k_1\mathbf{x}_1(0) + k_2\mathbf{x}_2(0) + k_3\mathbf{x}_3(0) + k_4\mathbf{x}_4(0)$. Notice that

$$\mathbf{x}_1(0) = \mathbf{v}_1$$

$$\mathbf{x}_2(0) = \mathbf{v}_2$$

$$\mathbf{x}_3(0) = \text{real part}(\mathbf{w})$$

$$\mathbf{x}_4(0) = \text{imaginary part}(\mathbf{w})$$

Using MATLAB

```
>> v1=[1 0 1 0]';v2=[1 -2 -2 1]';
>> V=[v1 v2 real(w) imag(w)]
```

V =

```
1.0000    1.0000    1.0000    1.0000
         0   -2.0000    0.0000   -1.0000
1.0000   -2.0000    1.0000         0
         0    1.0000    1.0000   -1.0000
```

```
>> x0=[1 -1 2 0]';
>> k=V\x0
```

k =

```
-9.0000
-2.0000
 7.0000
 5.0000
```

Hence $\mathbf{x}(t) = -9\mathbf{x}_1(t) - 2\mathbf{x}_2(t) + 7\mathbf{x}_3(t) + 5\mathbf{x}_4(t)$.