Minimal Immersions with Prescribed Boundary

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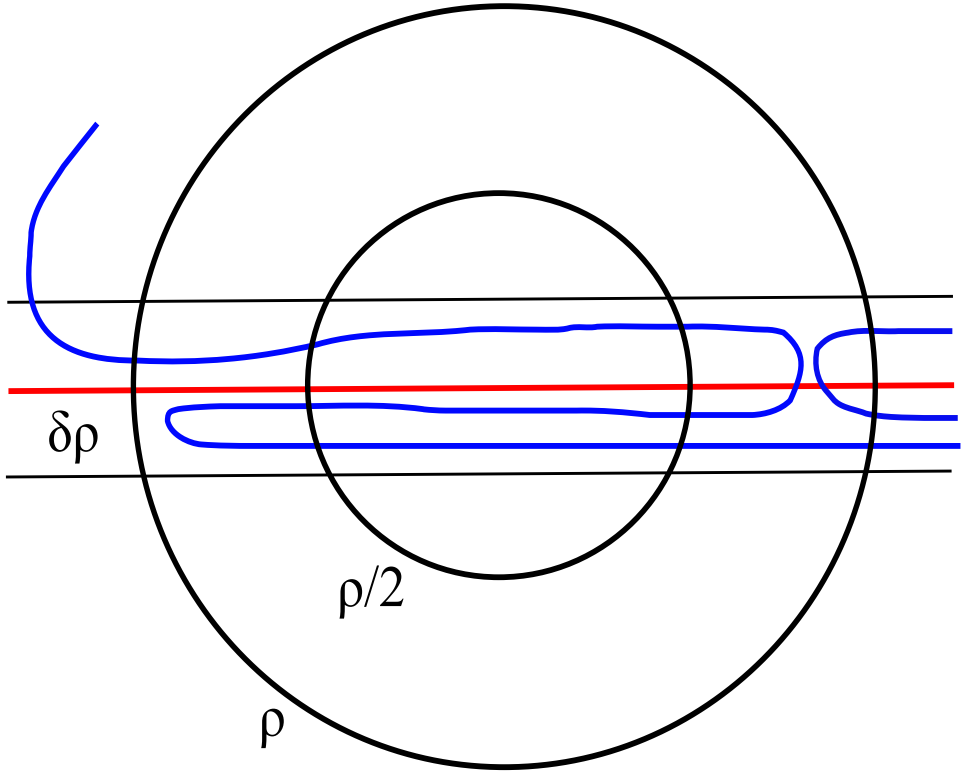
[SS81] Thm: $M^n \subset \mathbb{R}^{n+1}$ embedded stable, $\mathcal{H}^{n-2}(\text{sing}M) < \infty$



[W] Thm: M^n immersed, then near $\Theta_M < 3$

$$\text{sing}M = \textbf{sing}M_{\leq n-7} \cup \text{branch}M_{\leq n-2}.$$

[SW07] Thm: PDE method for producing stable immersions $G^n \subset \mathbb{R}^{n+1}$ with branch set $(n-2)$-dml $C^{1,\alpha}$ submanifold, $\mathbb{Z}_{\frac{2\pi}{k}}$ symmetric.

[R07] Thm: For $n = 2$:

- More complete description of the PDE method.
- Give "even" examples of branched $G \subset \mathbb{R}^3$.

I. PDE Method

$$\mathcal{F}_0(v) = \int_{\mathcal{D}} 4r^2 \sqrt{1 + \frac{|Dv|^2}{4r^2}}\ dx$$

$x = (x_1, x_2) = re^{i\theta}$ and $(x, t) \in \mathbb{R}^3$

$\mathcal{D} = \{|x| < 1\}$, $\mathcal{D}_\rho = \{|x| < \rho\}$

$$\mathcal{M}_0(v) = \sum_{j=1}^{2} D_{x_j} \left(\frac{D_{x_j} v}{\sqrt{1 + \frac{|Dv|^2}{4r^2}}} \right)$$

If

$$\mathcal{M}_0(u_0) = \sum_{j=1}^{2} D_{x_j} \left(\frac{D_{x_j} u_0}{\sqrt{1 + \frac{|Du_0|^2}{4r^2}}} \right) = 0$$

Then $u(r, \theta) = u_0(r^{\frac{1}{2}} e^{i\theta/2})$ for $r \in (0, 1), \theta \in \mathbb{R}$

- locally solution to the MSE
- period 4π

$\mathcal{M}_0$ = Two-Valued MSE =2MSE.

[SW07] Thm: For all $\varphi_0 \in C(S^1)$, there is a unique

$$u_0 \in C(\overline{\mathcal{D}} \setminus \{0\}) \cap C^\infty(\mathcal{D} \setminus \{0\})$$

- $\mathcal{M}_0(u_0) = \sum_{j=1}^2 D_{x_j} \left(\frac{D_{x_j} u_0}{\sqrt{1+\frac{|Du_0|^2}{4r^2}}} \right) = 0$

- $u_0|_{S^1} = \varphi_0$

- $\sup_{\mathcal{D}_\rho \setminus \mathcal{D}_\sigma} |Du_0| < C(\sigma, \rho, \sup_{S^1} |\varphi_0|)$

Proof:

- Let $r_\delta = \begin{cases} r & \text{if } r > \delta \\ \geq \delta/2 & \text{if } r < \delta \end{cases}$

$$\mathcal{M}_\delta(v) = \sum_{j=1}^{2} D_{x_j} \left(\frac{D_{x_j} v}{\sqrt{1 + \frac{|Dv|^2}{4r_\delta^2}}} \right) = 0.$$

- Interior and Boundary gradient estimates for Non-Uniformly Elliptic PDE's by [S76] $\rightarrow \exists u_\delta|_{S^1} = \varphi_0$.

- Some subsequence

$$u_\delta \rightarrow u_0$$

uniformly on compact subsets of $\mathcal{D} \setminus \{0\}$.

- Uniqueness follows as in [F53].

Note that:

- $u_0 \in C(\mathcal{D} \setminus \{0\})$

- Strong Maximum Principle for differences

$$\sup_{\mathcal{D}\setminus\{0\}} |u_0 - \tilde{u}_0| \leq \sup_{S^1} |\varphi_0 - \tilde{\varphi}_0|.$$

- We want branched $C^{1,\alpha}$ stable surfaces.

- For $\mathcal{M}_0(u_0) = 0$, let

$$G := \{(re^{i\theta}, u(r,\theta)) : r \in (0,1), \theta \in \mathbb{R}\}$$

with $u(r,\theta) = u_0(r^{\frac{1}{2}} e^{i\theta/2})$.

- Minimal in $(\mathcal{D} \times \mathbb{R}) \setminus (\{0\} \times \mathbb{R})$.

Stability depends on whether $u_0 \in C(\mathcal{D})$.

II. $\mathbb{Z}_{\frac{2\pi}{k}}$ Examples

[SW07] Thm: If $u_0 \in C(\mathcal{D})$, then G is $C^{1,\alpha}$ stable branched.

- For $\zeta = \zeta_0(r^{\frac{1}{2}} e^{i\theta/2})$

$$\int_G \mathrm{div}_G \zeta \; d\mathcal{H}^2 = 0$$

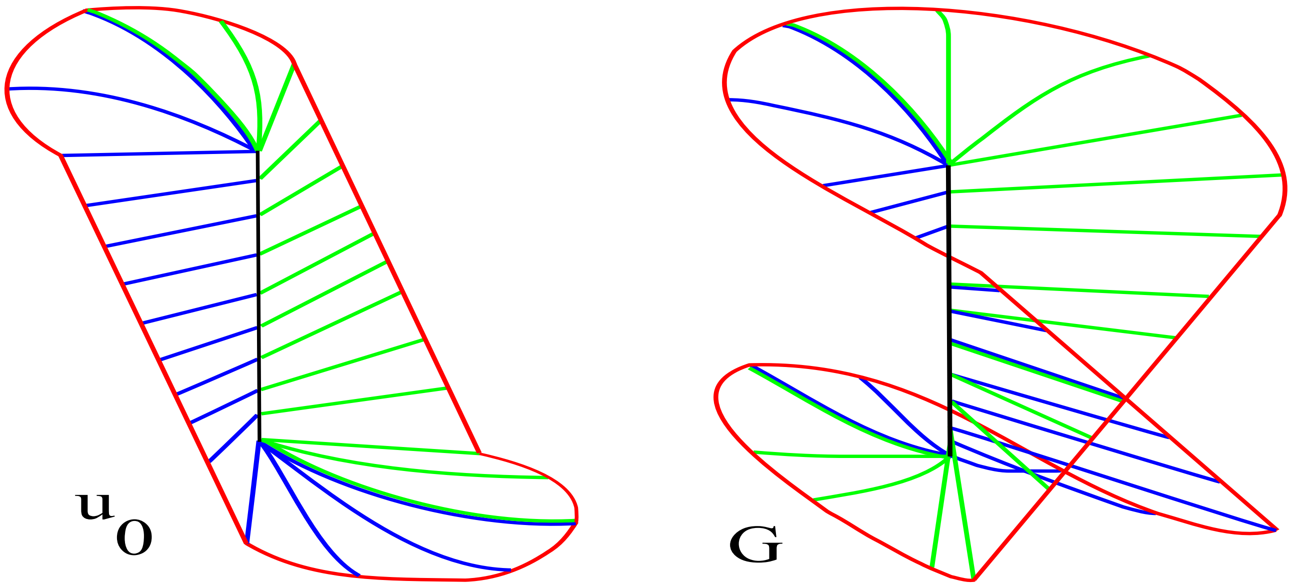
- Uniform Mass Bound: (for any G)

$$\mathcal{H}^2(G \cap B_\rho((x,t))) \leq c\rho^2$$

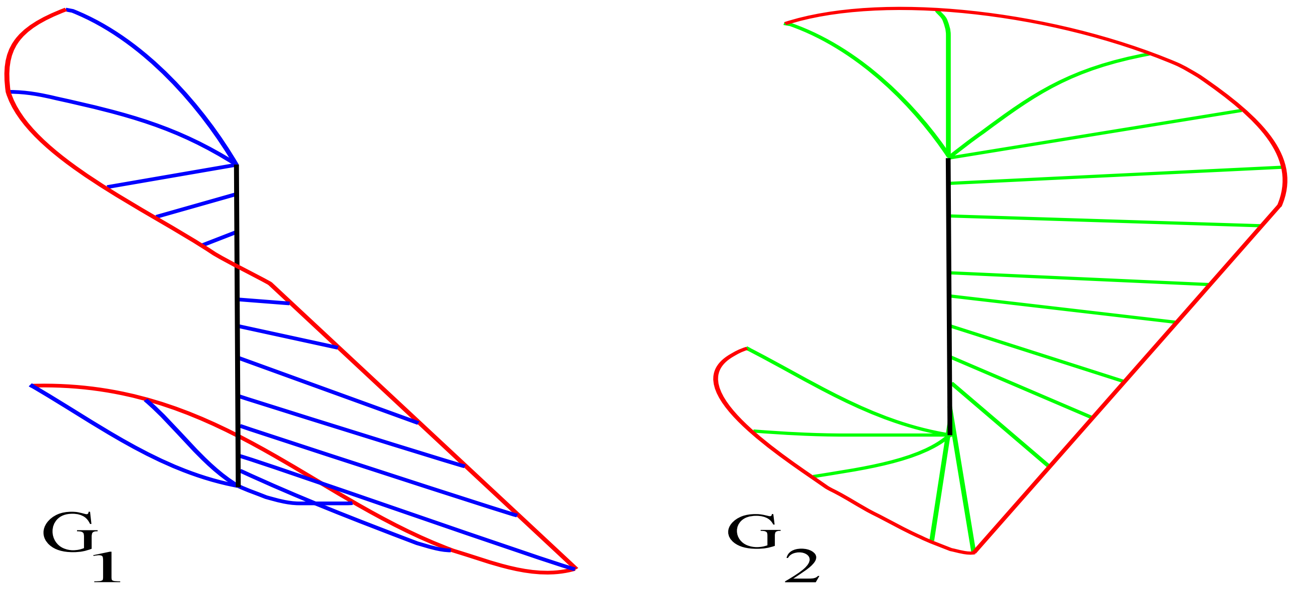
  if $B_{2\rho}((x,t)) \subset \mathcal{D} \times \mathbb{R}$.

[SW07] Thm: If $\varphi_0(\theta + \frac{2\pi}{k}) = \varphi_0(\theta)$ with $k > 1$ odd, then $u_0 \in C(\mathcal{D})$.

- $\overline{G} \cap (\{0\} \times \mathbb{R}) = \{0\} \times [\underline{\lim}_{r \to 0} u_0, \overline{\lim}_{r \to 0} u_0]$



interval of discontinuity $= (\underline{\lim}_{r \to 0} u_0, \overline{\lim}_{r \to 0} u_0)$



- For $\theta_0 \in [0, 2\pi)$ then

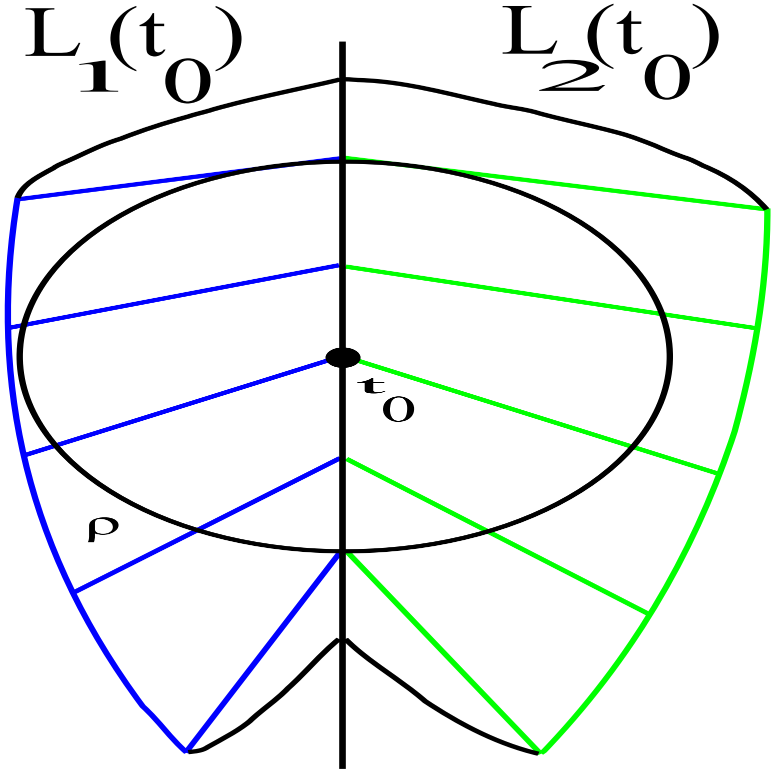
$$\begin{cases} u_1(r,\theta) = u_0(r^{\frac{1}{2}}, \theta/2) \\ u_2(r,\theta) = u_0(r^{\frac{1}{2}}, \theta/2 + \pi) \end{cases}$$

for $0 < r < 1$ and $\theta_0 < \theta < \theta_0 + 2\pi$.

- $G \cap ((\mathcal{D} \setminus \{re^{i\theta_0}\}) \times \mathbb{R}) = G_1 \cup G_2$.

[SW] Thm: If $u_0 \notin C(\mathcal{D})$, then at a dense set of "good" heights $t_0 \in (\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0)$

$G \cap B_\rho((0,t_0)) =$

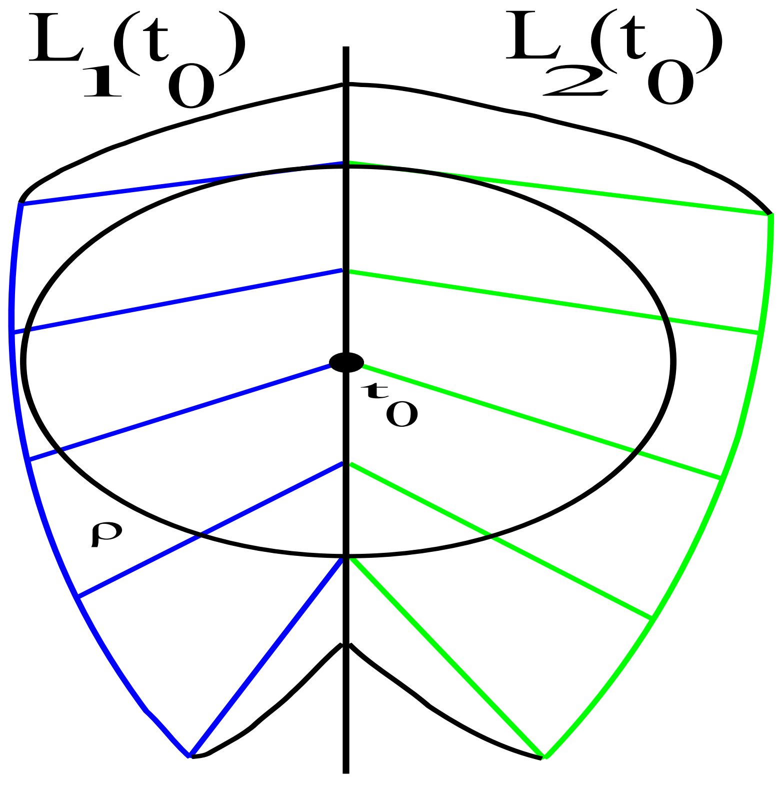


increasing/decreasing

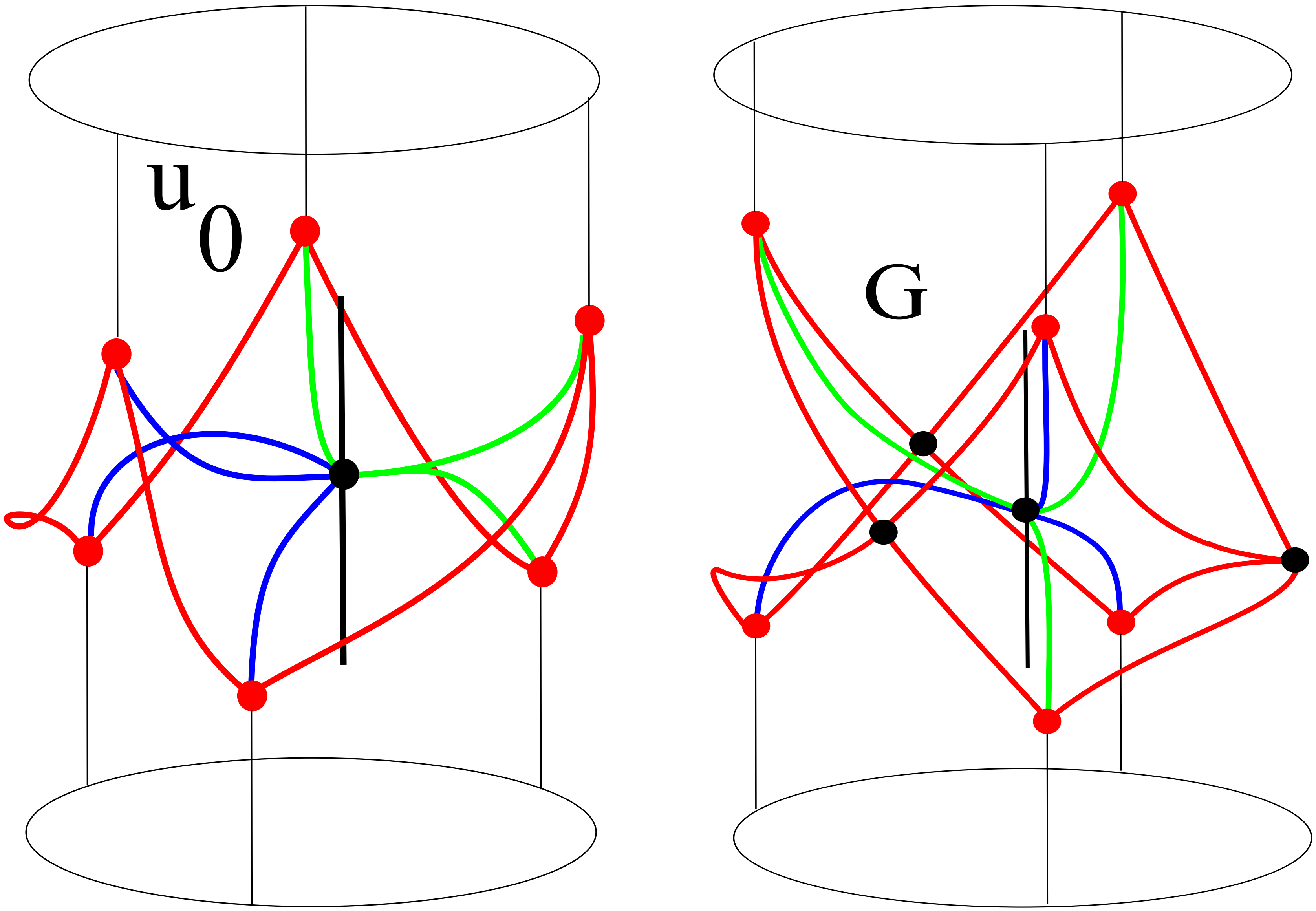
If $\varphi_0(\theta + \frac{2\pi}{k}) = \varphi_\theta(\theta)$:

- $u_0(r, \theta + \frac{2\pi}{k}) = u_0(r, \theta)$, and G is $\mathbb{Z}_{\frac{2\pi}{k}}$.

- But if $u_0 \notin C(\mathcal{D})$:



$\frac{2\pi}{3}$:



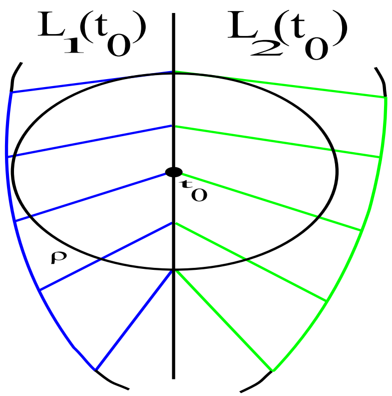
III. Even Examples

Thm: Suppose $\varphi_{s,0} \in C(S^1)$

- $\varphi_{s,0}(-\theta) = \varphi_{s,0}(\theta)$ even
- $\varphi_0 \to \varphi_{s,0} \to -\varphi_0$ continuously

then for some $s \in [0,1]$, $u_{s,0} \in C(\mathcal{D})$.

Thm: If $u_0 \notin C(\mathcal{D})$, then every t_0 is good:



Proof: φ_0 even, let $u_0 \notin C(\mathcal{D})$:

- $u_0(r, -\theta) = u_0(r, \theta)$

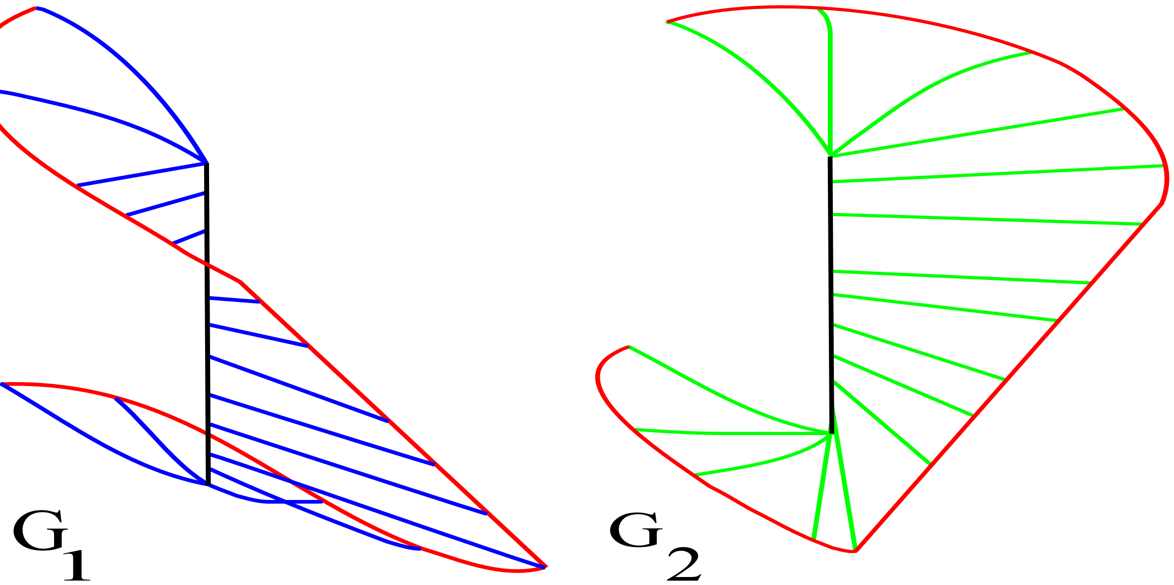
- Cut G along $x_1 > 0$:

$$\begin{cases} u_1(r,\theta) = u_0(r^{\frac{1}{2}}, \theta/2) \\ u_2(r,\theta) = u_0(r^{\frac{1}{2}}, \theta/2 + \pi) \end{cases}$$

  for $0 < r < 1$ and $0 < \theta < 2\pi$.

- Then $G_1 =$ reflection across x_1 of G_2 :

$$u_1(r, 2\pi - \theta) = u_2(r, \theta).$$

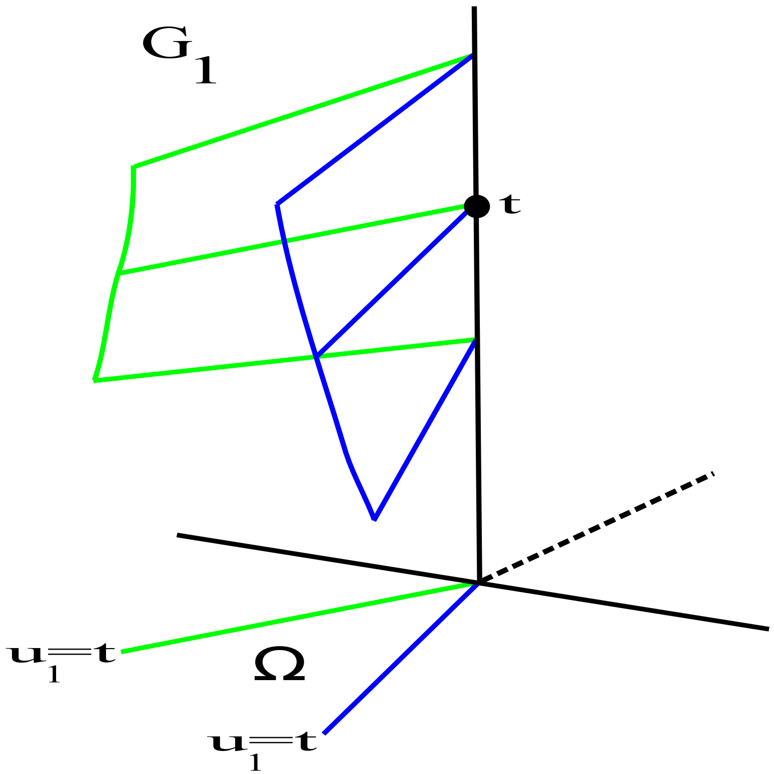


- If t is a good height (dense)

$$L_1(t) \subset G_1 \cap \{x_1 < 0\}$$

$$L_2(t) \subset G_2 \cap \{x_1 < 0\}.$$

- Suppose $L_1(t), L_2(t) \subset G_1 \cap \{x_1 < 0\}$

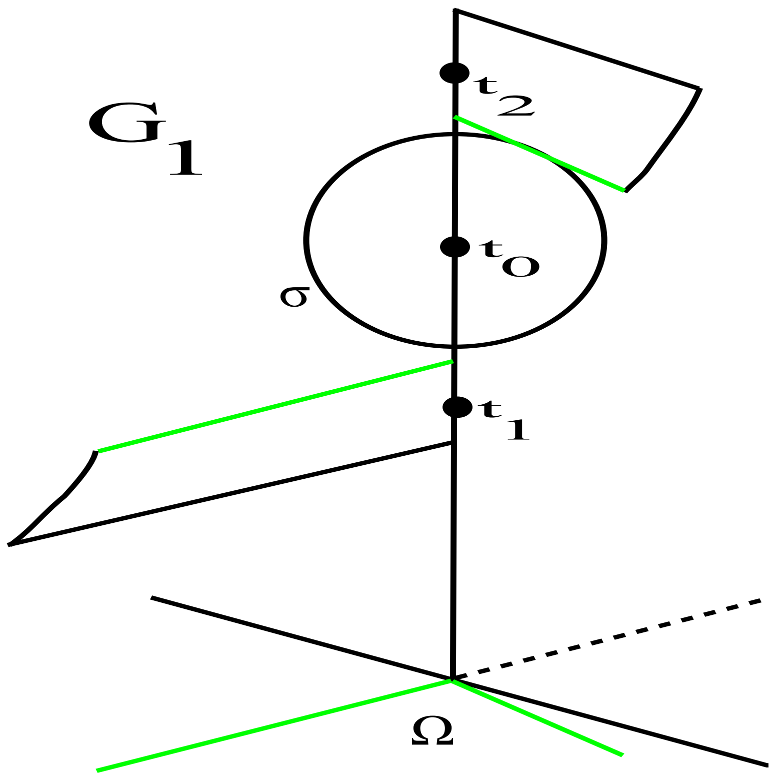


- Since $\Omega \subset \{x_1 < 0\}$, a barrier argument:

$$\lim_{x \to 0, x \in \Omega} u_1(x) = t$$

No!

- $t_0 \in (\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0)$, choose two good heights $t_1 < t_0 < t_2$.



- $\partial G_1 \cap B_\sigma((0,t_0)) = \{0\} \times (t_0 - \sigma, t_0 + \sigma)$

- $G_1 \cap B_\sigma((0,t_0)) = \text{graph}_\Omega\ u_1$

- Reflection Principle of [A75]

$$G_1 + R_{\#} G_1$$

is stationary in $B_\sigma((0, t_0))$.

- Monotonicity Formula holds for G_1 at $(0, t_0)$

$$\frac{\mathcal{H}^2(G_1 \cap B_\sigma((0, t_0)))}{\pi \sigma^2} \searrow \Theta_{G_1}((0, t_0)).$$

- G_1 stable, $\partial G_1 = \{0\} \times (t_0 - \sigma, t_0 + \sigma)$ in $B_\sigma((0, t_0))$

$$\mathbb{C} = \sum_{k=1}^{q_0} |\mathcal{H}_k|$$

by [SS81], with

$$\mathcal{H}_k \subset \{x_1 < 0\}.$$

- For σ small

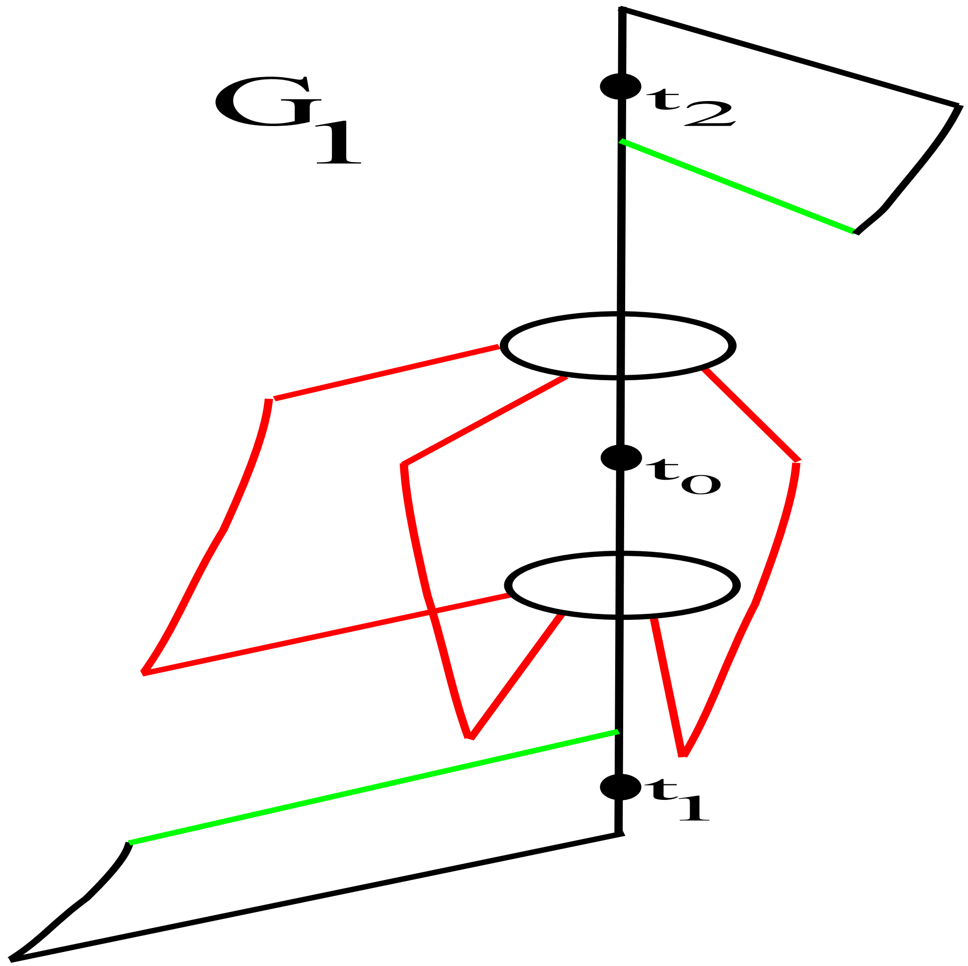
$$\text{dist}_{\mathcal{H}}(G_1 \cap B_\sigma((0,t_0)), \bigcup_{k=1}^{q_0} \mathcal{H}_k) < \delta\sigma,$$

and [SS81] says a stable surface close to a plane is a graph over that plane

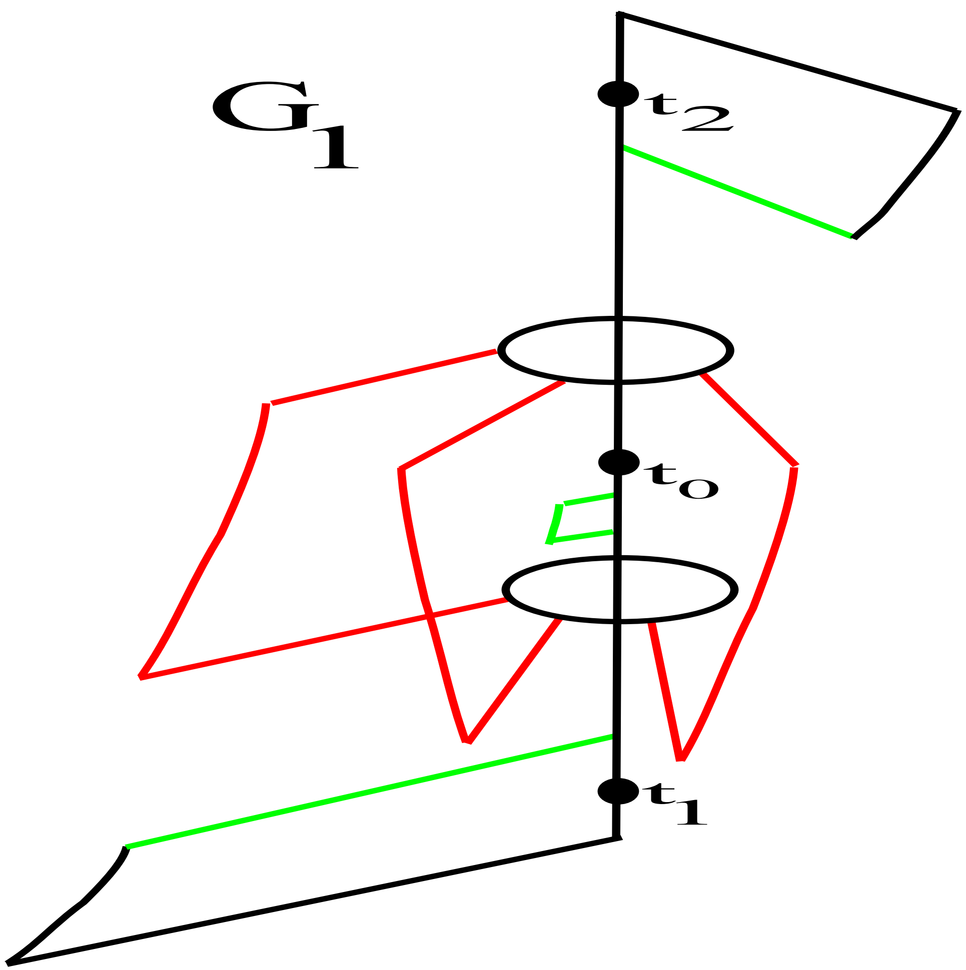
$$G_1 \cap B_\sigma((0,t_0)) \setminus (\mathcal{D}_{\delta\sigma} \times \mathbb{R}) = \cup_{k=1}^{q_0} \mathcal{L}_k$$

with

$$\mathcal{L}_k = \text{graph over } \mathcal{H}_k.$$



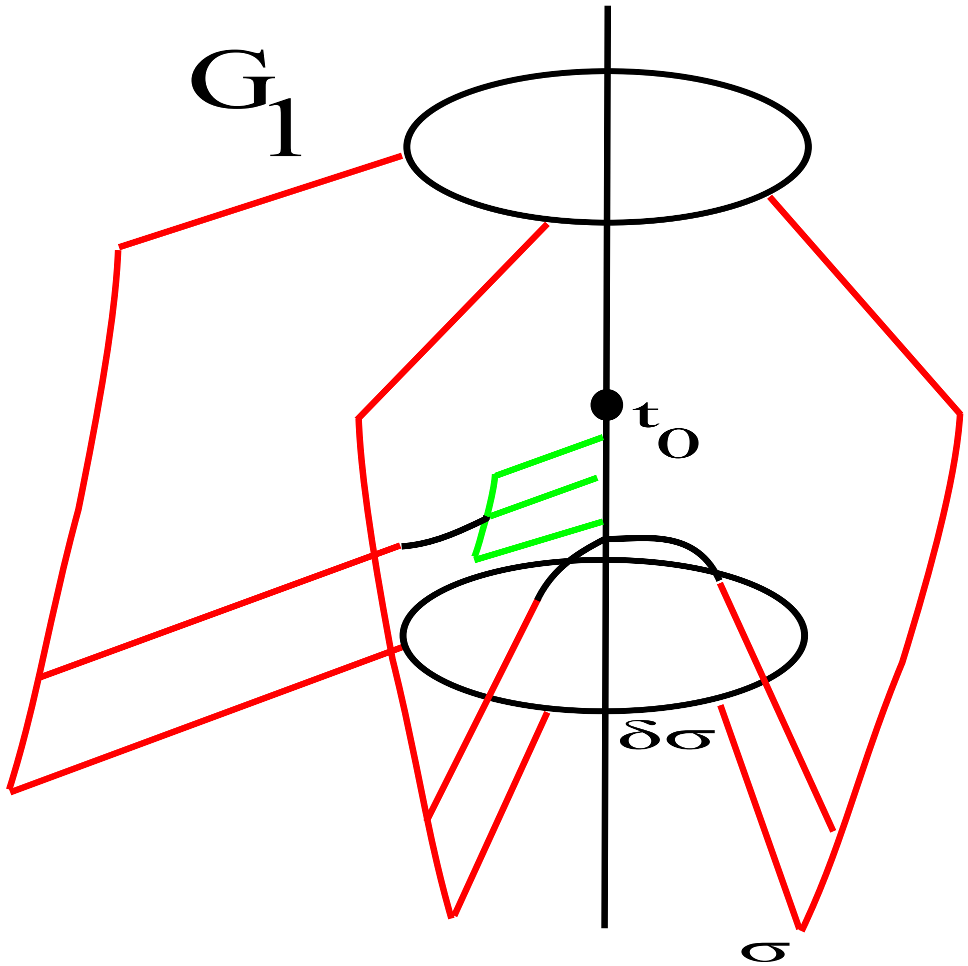
- By Allard's BR [A75], suppose $q_0 = 3$.

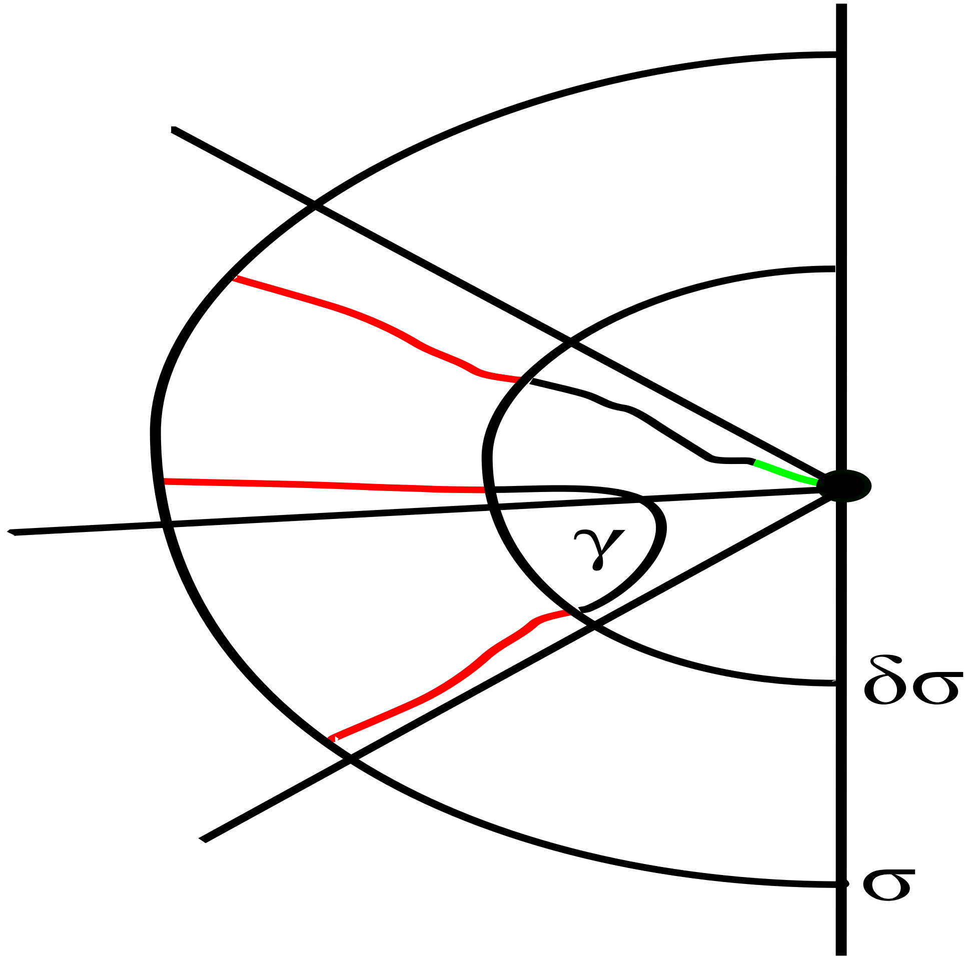


- $t \in (t_0 - \sigma, t_0)$ are good heights.

Applying Sard's Thm for $t_0 - \sigma < t < t_0$

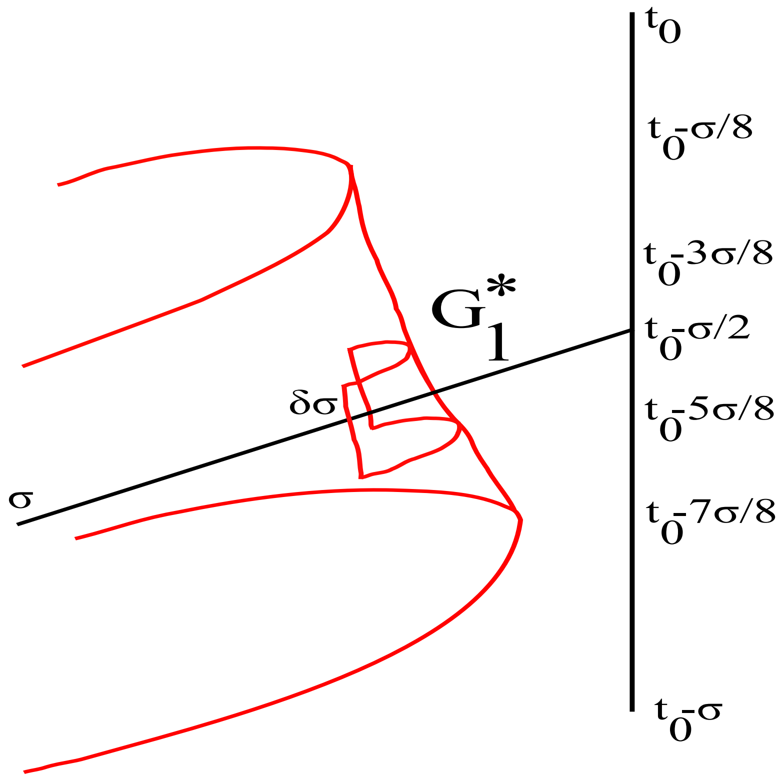
$G_1 \cap (\mathcal{D}_\sigma \times \{t\}) =$





$$c(\pi - \vartheta) \leq |\nu_{G_1}(\gamma(1)) - \nu_{G_1}(\gamma(0))| \leq \int_\gamma |A_{G_1}| \, d\mathcal{H}^1$$

We get $G_1^* \subset \mathcal{D}_{\delta\sigma} \times [t_0 - \frac{5\sigma}{8}, t_0 - \frac{3\sigma}{8}]$



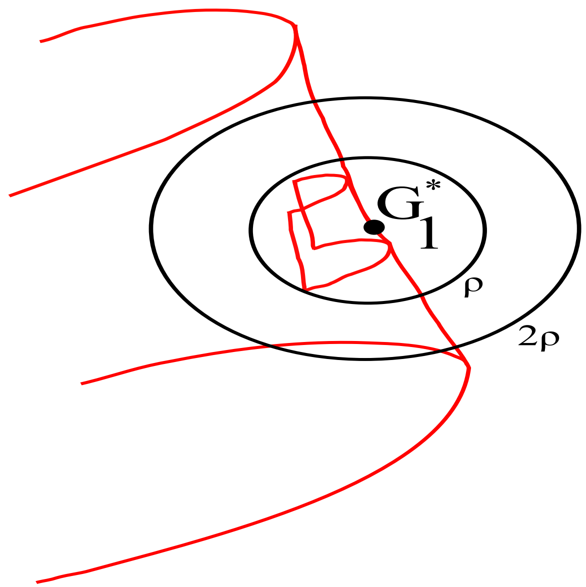
$$\frac{c(\pi - \vartheta)^2}{\delta} \leq \int_{G_1^*} |A_{G_1}|^2 \, d\mathcal{H}^2$$

[SS83] Thm: $M \subset B_1(0)$ simply connected, minimal, $0 \in M$ and $B_{2\rho}(0) \subset B_1(0)$:

$$\int_{M_\rho^*} |A_M|^2 \leq c_{13} \left(\frac{\mathcal{H}^2(M \cap B_{2\rho}(0))}{(2\rho)^2} + 1 \right)$$

$M_\rho^* =$ component of $M \cap B_\rho(0)$ with $0 \in M_\rho^*$.

- $\rho = \frac{3\sigma}{16}$



- [SS83] gives

$$\frac{c(\pi - \vartheta)}{\delta} \le c_{13} \left(\frac{\mathcal{H}^2(G_1 \cap B_{2\rho}(x^*, t_0 - \sigma/2))}{(2\rho)^2} + 1 \right) \le \text{Uniform Mass Bound}$$

- $\mathbb{C} =$ one vertical half-plane, so

$$G_1 \cap B_\rho((0, t_0)) = L_1(t_0).$$

Thm: If $u_0 \notin C(\mathcal{D})$, because the set of good heights are dense from [SW07],

$$G \cap B_\rho((0,t)) = L_1(t) \cup L_2(t)$$

every $t \in (\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0)$. $\square$

Thm: $\varphi_0 \to \varphi_{s,0} \to -\varphi_0$ even, then some $u_{s,0} \in C(\mathcal{D})$, and so G_s is $C^{1,\alpha}$ stable branched.

Proof:

- $\sup_{\mathcal{D}\setminus\{0\}} |u_{s,0} - u_{\tilde{s},0}| \leq \sup_{S^1} |\varphi_{s,0} - \varphi_{\tilde{s},0}|$

- $\exists \rho > 0$

$$(-\rho, \rho) \subset (\underline{\lim}_{r\to 0} u_{s,0}, \overline{\lim}_{r\to 0} u_{s,0})$$

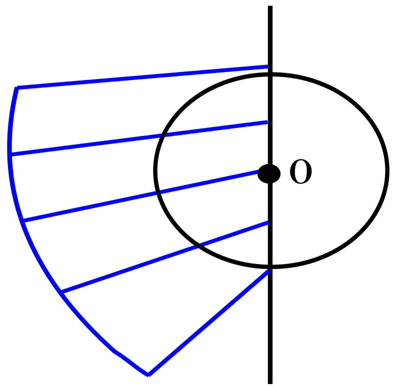
  if $u_{s,0} \notin C(\mathcal{D})$.

- Cut each $G_s = G_{s,1} \cup G_{s,2}$ along $x_1 > 0$.

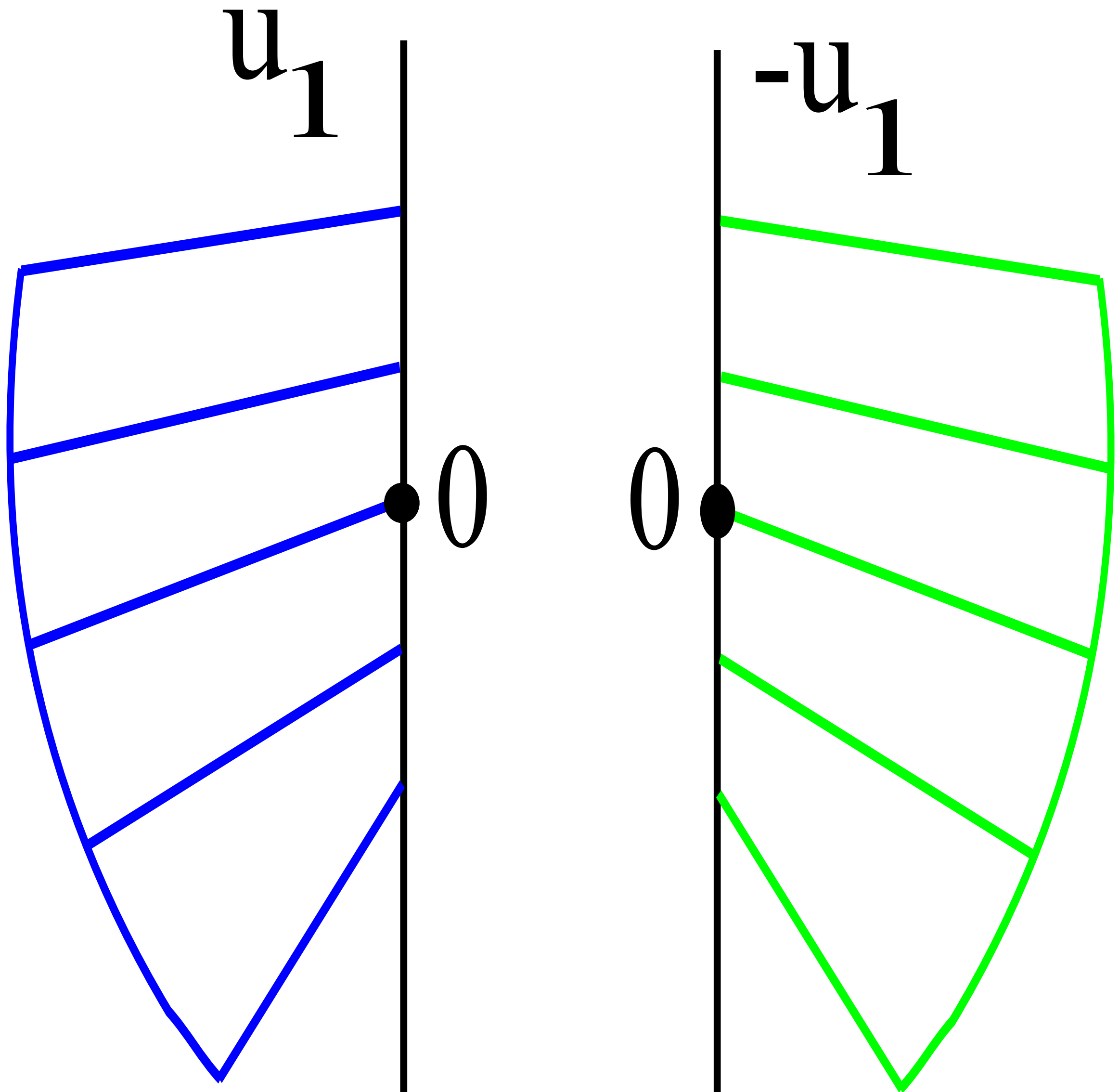
- By Allard's BR [A75], and

$$\sup_{s,\mathcal{D}_\rho \setminus \mathcal{D}_\sigma} |Du_{s,1}| \leq C(\sigma, \rho, \sup_{s,S^1} |\varphi_{s,0}|)$$

$G_{s,1} \cap B_\rho(0) =$



- $u_1 \to u_{s,1} \to -u_1$.



u_1
0
0
$-u_1$

IV: Miscellany

Proof: There is a dense set of good heights.

- $\kappa = \inf_{t\in[a,b]} \Theta_G((0,t))$ then for all ϵ, $\exists$ t_0, ρ_0, so that for all $t \in (t_0 - \rho_0, t_0 + \rho_0), \rho \le \rho_0$

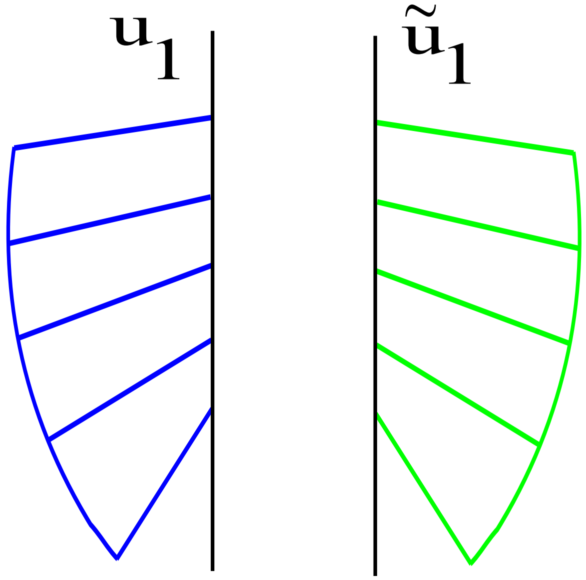
$$\kappa \le \frac{\mathcal{H}^2(G \cap B_\rho((0,t)))}{\pi\rho^2} < \kappa + \epsilon.$$

- [SS81] $\rightarrow$ $G \cap B_{\rho_0}((0,t_0)) = \cup_{j=1}^{q_0} L_j(t_0)$ with $q_0 =$ even.

- Barrier argument $\rightarrow$ $q_0 = 2$.

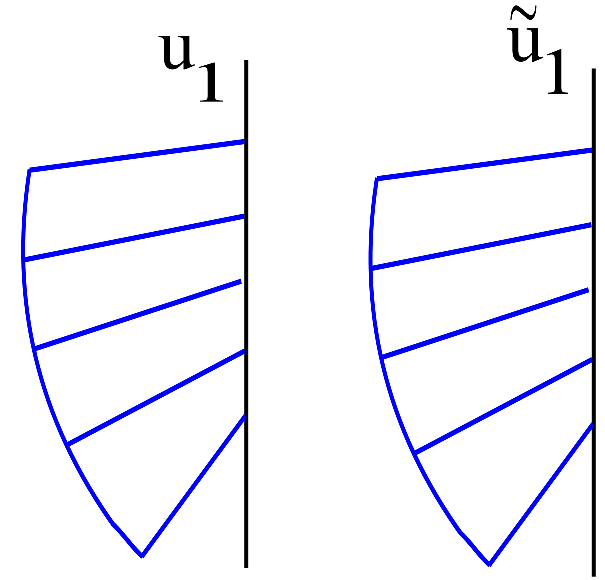
Instead of deforming $\varphi_0 \to -\varphi_0$

Thm: $\varphi_0 \to \varphi_{s,0} \to \tilde{\varphi}_0$ even, such that cutting along $\{x_1 > 0\}$



then some $u_{s,0} \in C(\mathcal{D})$.

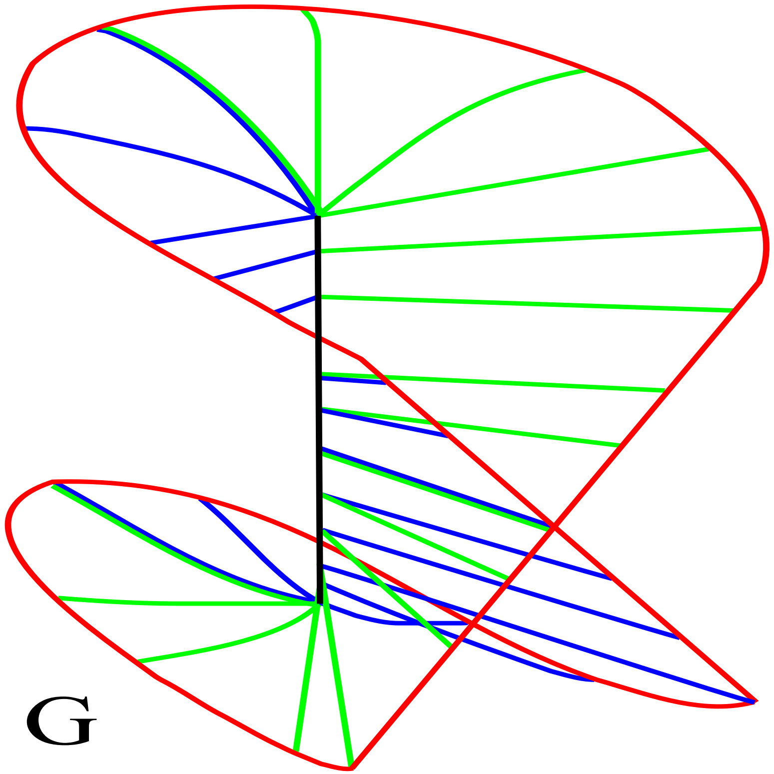
Conj: If φ_0 and $\tilde{\varphi}_0$ are even



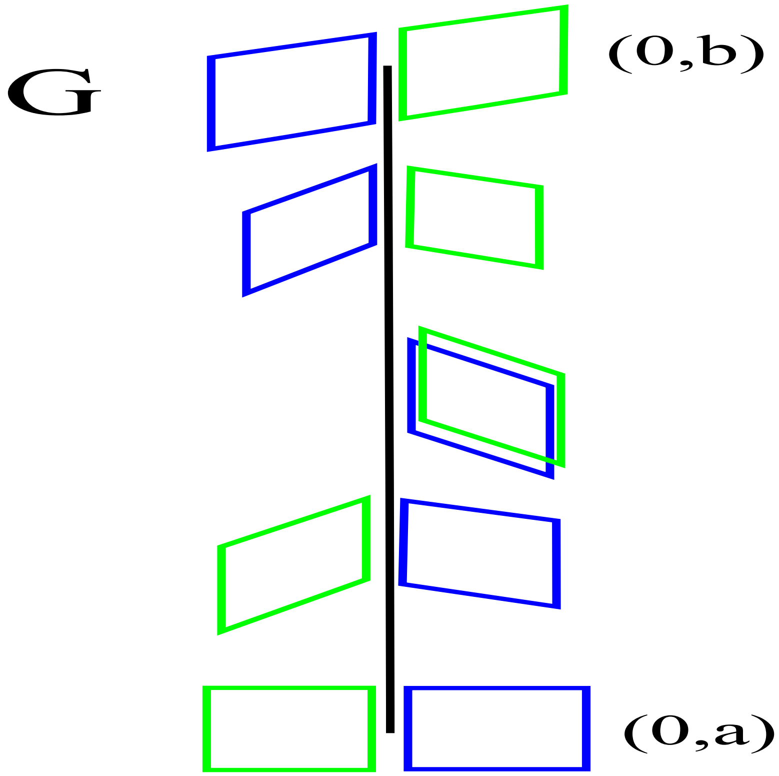
then we can find a deformation with $u_{s,0} \notin C(\mathcal{D})$.

Thm: The data $\varphi_0 = \cos(\theta)$ gives $u_0 \notin C(\mathcal{D})$.

and it looks like:



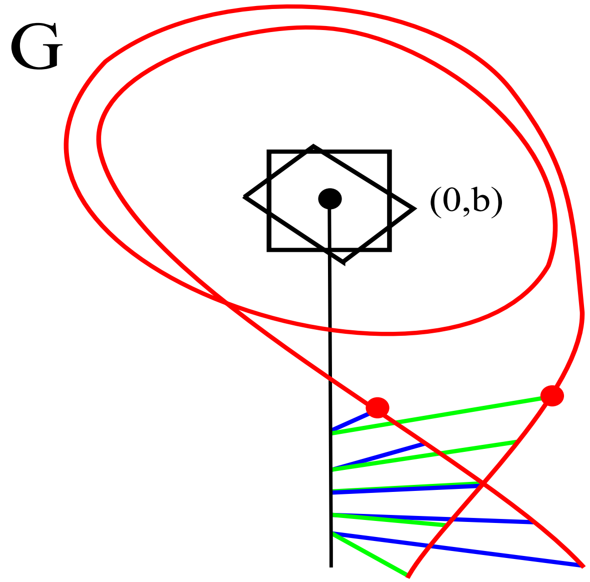
Thm: If $u_0 \notin C(\mathcal{D})$ then we have unique tangent cones at the endpoints, each of which is a vertical plane.



- $\bigcup_{t,j=1,2} H_j(t) \neq \mathbb{R}^3$

- Endpoints = Branch points for $G \cup R_{\#}G$.

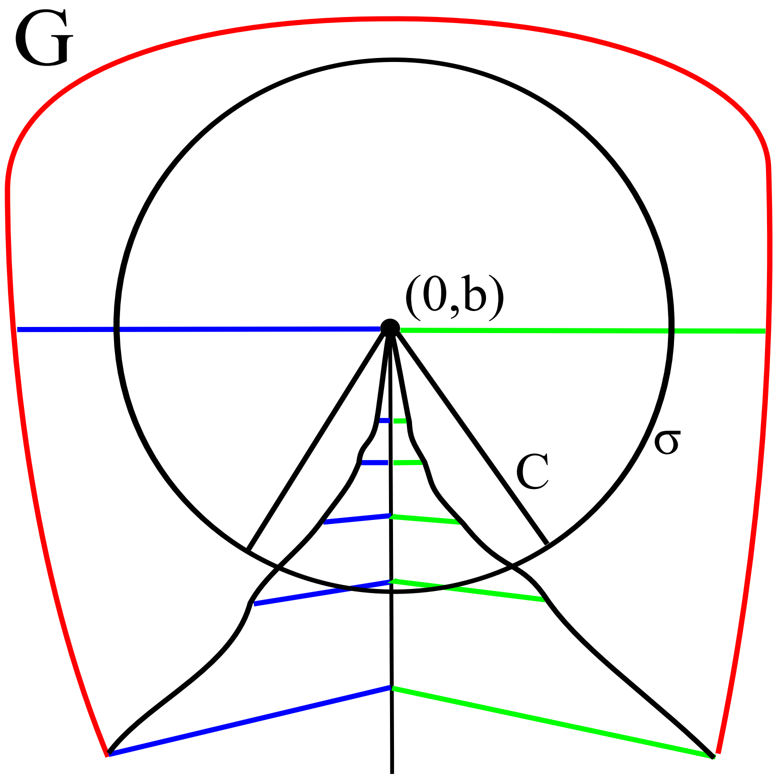
Proof:



- $\mathbb{C}$ = vertical half-planes, and is stationary.

- Barrier argument $\rightarrow < 3\pi$.

- [SS81], Sard's $\rightarrow \mathbb{C}$ = vertical plane.



- [SS81] $\to G \cap B_\sigma((0,b)) \setminus C =$ graph over $\mathbb{C}$.

- [SS83] $\to G_{blue} \cup R_\# G_{blue} =$ graph over $\mathbb{C}$.

- q-Valued MSE

$$\mathcal{M}_0(v) = \sum_{j=1}^{2} D_{x_j}\left(\frac{D_{x_j}v}{1+\frac{|Dv|^2}{(qr^{q-1})^2}}\right)$$

- $u_0 \notin C(\mathcal{D})$, then for all but finitely many $t \in (\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0)$

$$G \cap B_\rho((0,t)) = \bigcup_{j=1}^{q_0} L_j(t)$$

$2 \leq q_0 \leq 2q-2$ even.

- For $q = 3$, $\cos(2\theta)$ gives $u_0 \notin C(\mathcal{D})$.