# 1 Abstract

We study solutions to the Two-Valued Minimal Surface Equation (2MSE) over the punctured unit disk $\mathcal{D}$ in $\mathbb{R}^2$, introduced in [SW07] and more fully developed in [R07], which cannot be extended continuously across the origin. By proving comparison results which allow us to conclude that boundary data on $\mathcal{D}$ give discontinuous solutions to the 2MSE based on how the boundary data compares to data which is known to give a discontinuous solution to the 2MSE, we use certain "comparison solutions" found in [R07] in an approach analogous to [F63]. We are hence able to show that boundary data, satisfying a certain evenness condition, which is steep at two points of $\partial\mathcal{D}$ and relatively flat elsewhere yields solution to the 2MSE which cannot be extended continuously across the origin.

# 2 Introduction

In [SW07] a PDE method producing examples of stable branched immersed minimal surfaces is introduced. In the case more fully developed by the author in [R07], this method involves solving a PDE which we call the Two-Valued Minimal Surface Equation (2MSE) over $\mathcal{D} \setminus \{0\}$ with $\mathcal{D} = \{x \in \mathbb{R}^2 : |x| < 1\}$. A main result of [SW07] states that if $u_0$ a solution to the 2MSE over the punctured unit disc can be extended continuously at the origin, then $\{(re^{i\theta}, u_0(r^{1/2}e^{i\theta/2}) : r \in (0,1), \theta \in [0, 4\pi)\}$ is a stable branched immersed minimal surface in $\mathcal{D} \times \mathbb{R}^3$, and is $C^{1,\alpha}$ at the origin. The proof of this involves modifying the methods to show single-valued solutions to the Minimal Surface Equation (MSE) satisfy $C^{1,\alpha}$ estimates to the present two-valued setting.

The present work complements the work of [SW07], with analogy to the historical development of the theory of the MSE. Here we focus on solutions to the 2MSE which cannot be extended continuously across the origin, and give conditions on the boundary data $\varphi_0 \in C(\partial\mathcal{D})$ so that the solution $u_0$ to the 2MSE in $\mathcal{D}$ with boundary data $\varphi_0$ cannot be extended continuously across the origin. We do this through the use of comparison surfaces in an approach similar to the one found in [F63]. There, Finn constructs families of "comparison" solutions to the MSE on quadrilaterals inscribed in $\mathcal{D}$, with discontinuous boundary data $-M, M$ for $M > 0$. Finn then shows that the gradient of the comparison solution at the origin bounds the gradient of any solution to the MSE on $\mathcal{D}$ with boundary data bounded by $M$ in absolute value and common value at the origin.

Presently, we use solutions to the 2MSE with boundary data $\varphi_{\alpha,0}(\theta) = \alpha\cos(\theta)$, which were shown by the author in [R07] to have solutions to the 2MSE which cannot be extended continuously across the origin. Through the use of Maximum Principle arguments, we essentially show that the gradient near the origin of solutions to the 2MSE with boundary data $P_0$ satisfying a certain evenness condition and "steeper" than that of a comparison solution $\varphi_{\alpha,0}$ must exhibit similar asymptotic behavior to that of the gradient near the origin of the solution to the 2MSE with boundary data $\varphi_{\alpha,0}$.

First, in §3 we discuss the results of [SW07] and [R07] we shall need. In §4 we state our main results. The most important result, Theorem 5.2, is a comparison principle which establishes that if $\varphi_0(e^{i\theta})$ and $P_0(e^{i\theta})$ are both boundary data even in $\theta$ such that the solution to the 2MSE $u_0$ with boundary data $\varphi_0$ cannot be extended continuously across the origin, then we cannot have that the solution $U_0$ to the 2MSE can be extended continuously across the origin with $U_0(0) \in [\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0]$. Theorem 5.2 is analogous to Theorem III.3 of [F63], and we prove Theorem 5.2 in §5.

In §7 we give the proof of Theorem 5.3, which is Conjecture 9.2 or [R07]. Theorem 5.3 states that the set of even boundary data which yield solutions to the 2MSE which cannot be extended continuously across the origin can be decomposed into two open path connected subsets of $C(\partial\mathcal{D})$. Next, §8 is devoted to showing that even boundary data which is sufficiently steep indeed have solution to the 2MSE which cannot be extended continuously across the origin. Finally, in §10 we extend our main results to the case of the $q$-valued Minimal Surface Equation, for $q \geq 2$.

# 3 Notation

Before we proceed, we list some notation conventions we shall use:

- We shall denote a point in $\mathbb{R}^3$ by $(x, t)$ where $x = (x_1, x_2) \in \mathbb{R}^2$ and $t \in \mathbb{R}$. We may also write $x \neq 0$ in polar coordinates by taking $x = re^{i\theta}$ with $r = |x|$, $\theta \in \mathbb{R}$.

- $B_\rho(x, t)$ will be the open ball in $\mathbb{R}^3$ centered at $(x, t)$ with radius $\rho$.

- $\mathcal{D}$ will be the open unit disc centered at the origin in $\mathbb{R}^2$. Also, $\mathcal{D}_\rho$ will denote the open disc with radius $\rho$ centered at the origin in $\mathbb{R}^2$.

- $\mathcal{H}^n$ will denote the $n$-dimensional Hausdorff measure in $\mathbb{R}^3$, for $n = 1, 2$.
- We denote $\Omega_0 \subset \mathcal{D}$ to be the region

$$\Omega_0 = \{re^{i\theta} : r \in (0,1), \theta \in (0, 2\pi)\}.$$

  Given a function $u_1 : (0,1) \times (0, 2\pi) \to \mathbb{R}$, we may take $u_1 : \Omega_0 \to \mathbb{R}$ by setting $u_1(x) = u_1(r, \theta)$ where $x = re^{i\theta}$ for $r \in (0,1), \theta \in (0, 2\pi)$.
- We shall make the abbreviation

$$\{\theta = \theta_0\} = \{(re^{i\theta_0}, t) : r > 0,\ t \in \mathbb{R}\}$$

  for the open vertical half-plane with angle $\theta_0$.

# 4 Preliminaries

We say that a function $u_0 \in C^\infty(\mathcal{D} \setminus \{0\})$ is a solution to the Two-Valued Minimal Surface Equation (2MSE) if

$$\mathcal{M}_0(u_0) = \sum_{j=1}^{2} D_{x_j} \left( \frac{D_{x_j} u_0}{\sqrt{1 + \frac{|Du_0|^2}{4r^2}}} \right) = 0.$$

The significance of the $4r^2$ term is that if $u_0$ is a solution to the 2MSE, then the function

$$u(r, \theta) = u_0(r^{\frac{1}{2}} e^{i\theta/2})$$

is locally in $r \in (0,1)$ and $\theta \in \mathbb{R}$ a solution to the Minimal Surface Equation (MSE) with period $4\pi$ in $\theta$. Existence of solutions to the 2MSE is established in [SW07].

**Theorem 4.1** *Given $\varphi_0 \in C(\partial\mathcal{D})$, there exists a unique solution $u_0 \in C^\infty(\mathcal{D} \setminus \{0\}) \cap C(\overline{\mathcal{D}} \setminus \{0\})$ to the 2MSE with boundary data $u_0|_{\partial\mathcal{D}} = \varphi_0$. Furthermore, for any $0 < \sigma < \rho < 1$ we have*

$$\|u_0\|_{C^k(\mathcal{D}_\rho \setminus \mathcal{D}_\sigma)} \leq C(\sigma, \rho, k, \|\varphi_0\|_{C(\partial\mathcal{D})}).$$

Uniqueness is shown in [R07] by following the argument that (single-valued) solutions to the MSE are unique (see [F53]). The following Maximum Principle is established in [R07]:

**Theorem 4.2** *If $u_0 \in C(\overline{\mathcal{D}} \setminus \{0\})$ is a (non-constant) solution to the 2MSE and $O$ is an open subset of $\mathcal{D}$, then $u_0$ cannot achieve its maximum or minimum in $O$. If $\tilde{u}_0 \in C(\overline{\mathcal{D}} \setminus \{0\})$ is also a solution to the 2MSE, then $u_0 - \tilde{u}_0$ cannot achieve its maximum or minimum in $O$.*

Given a solution $u_0$ to the 2MSE, we let $u(r, \theta) = u_0(r^{\frac{1}{2}} e^{i\theta/2})$ and

$$G = \{(re^{i\theta}, u(r, \theta)) : r \in (0, 1), \theta \in [0, 4\pi)\}.$$

$G$ is therefore an immersed minimal surface in $\mathcal{D} \times \mathbb{R} \setminus (\{0\} \times \mathbb{R})$ which we call the two-valued graph corresponding to $u_0$. The two-valued graph $G$ of $u_0$ can be decomposed into the union of two single-valued minimal graphs, if we allow the introduction of a boundary. To do this, consider $\Omega_0 = \{re^{i\theta} : r \in (0, 1), \theta \in (0, 2\pi)\}$ and define

$$\begin{cases} u_1(r, \theta) = u_0(r^{1/2}, \theta/2) \\ u_2(r, \theta) = u_0(r^{1/2}, \theta/2 + \pi) \end{cases}$$

for $r \in (0, 1)$ and $\theta \in (0, 2\pi)$. Then

$$G \cap (\Omega_0 \times \mathbb{R}) = G_1 \cup G_2$$

where

$$G_j = \{(re^{i\theta}, u_j(r, \theta)) : r \in (0, 1), \theta \in (0, 2\pi)\}.$$

We call $u_1, u_2$ the component functions of $u_0$.

Although we are not guaranteed that solutions to the 2MSE can be extended continuously across the origin, solutions which can be extended continuously across the origin are of significant importance. The following is a main result of [SW07] to this end:

**Theorem 4.3** *Suppose $u_0 \in C(\overline{\mathcal{D}} \setminus \{0\})$ is a solution to the 2MSE which can be extended continuously across the origin. Then $G$ is a $C^{1,\alpha}$ stable branched immersed minimal surface.*

First, $G$ is $C^{1,\alpha}$ in the sense that $u$ is $C^{1,\alpha}$ as a two-valued function. This means that for every $\sigma \in (0, 1)$ and $\theta_0 \in (0, 4\pi)$, there is an $\alpha = \alpha(\|\varphi_0\|_{C(\partial\mathcal{D})}, \sigma) \in (0, 1)$ and a constant $C = C(\sigma, \|\varphi_0\|_{C(\partial\mathcal{D})})$ such that for the (single-valued) function $u(r, \theta) = u_0(r^{1/2}, \theta/2)$ for $r \in (0, 1), \theta \in (\theta_0, \theta_0 + 2\pi)$, $Du$ is uniformly Hölder continuous in the region

$$\{re^{i\theta} : r \in (0, 1 - \sigma), \theta \in (\theta_0 + \sigma, \theta_0 + 2\pi - \sigma)\}$$

with Hölder exponent $\alpha$ and Hölder norm bounded by $C$.

Second, $G$ is stable in the sense that the stability inequality

$$\int_G \zeta^2 |A_G|^2 \; d\mathcal{H}^2 \leq \int_G |\nabla^G \zeta|^2 \; d\mathcal{H}^2$$

holds for any $\zeta \in C^1((\mathcal{D} \times \mathbb{R}) \setminus (\{0\} \times \mathbb{R}))$ vanishing near $\partial\mathcal{D} \times \mathbb{R}$, where $A_G$ is the second fundamental form of $G$.

Here we study solutions to the 2MSE which cannot be extended continuously across the origin. In particular, we study solutions with continuous boundary data even in $\theta$, that is data such that $\varphi_0(-\theta) = \varphi_0(\theta)$ if we write $\varphi_0(\theta) = \varphi_0(e^{i\theta})$. Uniqueness implies that solutions to the 2MSE with even boundary data are also even in $\theta$. One can check then that the two-valued graph corresponding to such a solution is symmetric under the reflection across the $x_1$-axis.

The following is a main result of [R07]:

**Theorem 4.4** *Suppose $\varphi_0 \in C(\partial\mathcal{D})$ is even boundary data and that the solution $u_0 \in C(\overline{\mathcal{D}} \setminus \{0\})$ to the 2MSE with boundary data $\varphi_0$ cannot be extended continuously across the origin. Then for every $t \in (\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0)$, there is a $\rho > 0$ such that*

$$G \cap B_\rho((0,t)) = L_1(t) \cup L_2(t)$$

*where each $L_j(t)$ is a smooth embedded minimal surface-with-boundary $\{0\} \times (t-\rho, t+\rho)$, with $L_j(t) \subset \{(x,t) \in \mathbb{R}^3 : x_1 < 0\}$. Furthermore, if $\Omega_0 = \{re^{i\theta} : r \in (0,1), \theta \in (0, 2\pi)\}$, then for the component functions $u_1, u_2$ of $u_0$ we can write*

$$L_1(t) \subset G_1, \;\; L_2(t) \subset G_2$$

*where $G_j = \operatorname{graph}_{\Omega_0} u_j = \{(re^{i\theta}, u_j(r,\theta)) : re^{i\theta} \in \Omega_0\}$.*

Note that if $u_0 \in C(\overline{\mathcal{D}} \setminus \{0\})$ is a solution to the 2MSE which cannot be extended continuously across the origin, then by continuity of $u_0$ in $\overline{\mathcal{D}} \setminus \{0\}$

$$\overline{G} \cap (\{0\} \times \mathbb{R}) = \{0\} \times [\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0].$$

We call $(\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0)$ the interval of discontinuity of $u_0$.

For each $t \in (\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0)$ and $j = 1, 2$, let $H_j(t)$ be the (open) tangent vertical half-plane of $L_j(t)$ at $(0,t)$. Since each $L_j(t)$ is contained in the graph of a function defined over $\Omega_0$ as well as, for $\rho > 0$ sufficiently small, is a graph over $H_j(t)$, then we have that $\frac{\partial u_j}{\partial \theta} \neq 0$ for each $re^{i\theta} \in \mathrm{proj}_{\Omega_0} L_j(t)$. We say that $L_j(t)$ is increasing if $\frac{\partial u_j}{\partial \theta} > 0$ for $re^{i\theta} \in \mathrm{proj}_{\Omega_0} L_j(t)$ and decreasing if $\frac{\partial u_j}{\partial \theta} < 0$ for $re^{i\theta} \in \mathrm{proj}_{\Omega_0} L_j(t)$. Continuity of $u_0$ in $\mathcal{D} \setminus \{0\}$ implies that the surfaces $L_j(t)$ cannot be both increasing or both decreasing, and that the choice as to whether $L_j(t)$ is increasing or decreasing is uniform in $t \in (\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0)$.

For each tangent vertical half-plane $H_j(t)$, we can choose $\theta_j(t) \in (\pi/2, 3\pi/2)$ such that $H_j(t) = \{\theta = \theta_j(t)\}$. The angles $\theta_j(t)$ vary continuously strictly monotonically in $t$. If the surface $L_j(t)$ is increasing, then the angles $\theta_j(t)$ are increasing as $t \in (\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0)$ increases, and if $L_j(t)$ is decreasing then the angles $\theta_j(t)$ are decreasing.

We can define an indexing function

$$\Phi : \{\varphi_0 \in C(\partial\mathcal{D}) : \varphi_0 \text{ is even}\} \to \{0, \pm 1\}.$$

First, if $\varphi_0 \in C(\partial\mathcal{D})$ is even boundary data such that the solution $u_0$ to the 2MSE with boundary data $\varphi_0$ can be extended continuously across the origin, then we define $\Phi(\varphi_0) = 0$. On the other hand, if the solution $u_0$ cannot be extended continuously across the origin, then we define $\Phi(\varphi_0) = -1$ if the surfaces $L_1(t) \subset G_1$ are decreasing. Otherwise, if the surfaces $L_1(t)$ are increasing, define $\Phi(\varphi_0) = 1$.

At the endpoints of the interval of discontinuity of $u_0$, [R07] gives an asymptotic description of the two-valued graph $G$ corresponding to $u_0$. Write the vertical plane $\{x_1 = 0\} = \{(0, x_2, t) : x_2, t \in \mathbb{R}\}$, and let $b = \overline{\lim}_{r\to 0} u_0$.

**Theorem 4.5** *Suppose $\varphi_0 \in C(\partial\mathcal{D})$ is even boundary data with $\Phi(\varphi_0) \neq 0$. Then for all sufficiently small $\delta > 0$, there is a $\rho > 0$ and a function*

$$w : \{x_1 = 0\} \cap B_\rho((0, b)) \to \mathbb{R}$$

*such that*

$$\overline{G} \cap B_\rho((0,b)) = \{(w((0,x_2,t)), x_2, t) : (0, x_2, t) \in B_\rho((0,b))\} \cap B_\rho((0,b)),$$

*where $w \in C(\{x_1 = 0\} \cap B_\rho((0,b)))$, $w((0,t)) = 0$ for $t \in (b-\rho, b]$, and $w$ is a smooth solution to the MSE with scaled $C^1$ norm less that $\delta$ :*

$$\frac{|w|}{\rho} + |Dw| < \delta$$

*in $(\{x_1 = 0\} \cap B_\rho((0,b))) \setminus \{(0,t) \in \mathbb{R}^3 : t \in (b-\rho, b]\}$.*

Since $G \cap B_\rho((0,b))$ is both a graph of a function defined over the vertical plane $\{x_1 = 0\}$ and the horizontal plane $\mathbb{R}^2$, then $w((0,x_2,t))$ must be either strictly increasing or strictly decreasing in $t$ for all $(0,x_2,t) \in B_\rho((0,b))$ with $x_2 > 0$. Thus, if $\Phi(\varphi_0) = -1$, that is if the surfaces $L_1(t)$ are decreasing, then $w((0,x_2,t))$ is strictly decreasing in $t$.

It is likewise true that at $a = \underline{\lim}_{r\to 0} u_0$ we can write $G$ as the graph off the vertical plane $\{x_1 = 0\}$ with scaled $C^1$ norm less than $\delta$. We therefore have that if $\Phi(\varphi_0) = -1$, then the angles $\theta_1(t)$ decrease from $3\pi/2$ to $\pi/2$ as $t$ increases from $a$ to $b$, while the angles $\theta_2(t)$ increase from $\pi/2$ to $3\pi/2$ (see Theorem 6.3 of [R07]).

In the language of the Theory of Varifolds (see [A72],[S83]), the above theorem states that as a two-dimensional varifold $G$ has unique tangent cone at $(0,b)$ and $(0,a)$ consisting of the vertical plane $\{x_1 = 0\}$ with multiplicity one.

Of particular interest is boundary data of the form $\varphi_{\alpha,0} = \alpha\cos(\theta)$ with $\alpha > 0$. An argument using the Hopf Boundary Point Lemma (see Theorem 10.1 of [R07]) establishes the following:

**Theorem 4.6** *Suppose that $\varphi_0 \in C(\partial\mathcal{D})$ is even, so $\varphi_0(-\theta) = \varphi_0(\theta)$, and $\varphi_0$ has the odd symmetry $\varphi_0(\theta+\pi) = -\varphi_0(\theta)$. If $\varphi_0(\theta) > 0$ for $\theta \in (-\pi/2,\pi/2)$, then $\Phi(\varphi_0) = -1$. Also, $\underline{\lim}_{r\to 0} u_0 = -\underline{\lim}_{r\to 0} u_0$ where $u_0$ is the solution to the 2MSE with boundary data $\varphi_0$.*

**Definition 4.7** *We call $\varphi_0 \in C(\partial\mathcal{D})$ cosine-like if $\varphi_0$ is even and has the odd symmetry $\varphi_0(\theta+\pi) = -\varphi_0(\theta)$ and $\varphi_0(\theta) > 0$ for $\theta \in (-\pi/2,\pi/2)$.*

That $\Phi(\varphi_0) \neq 0$ for cosine-like boundary data follows from Theorem 10.1 of [R07], and we show $\Phi(\varphi_0) = -1$ specifically in Lemma 7.1. The identity $\underline{\lim}_{r\to 0} u_0 = -\underline{\lim}_{r\to 0} u_0$ follows because, by uniqueness of solutions to the 2MSE, we have $u_0(r,\theta+\pi) = -u_0(r,\theta)$.

# 5 Main Results

Our main result is a comparison principle. We first state a definition:

**Definition 5.1** *Suppose $\varphi_0, P_0 \in C(\partial\mathcal{D})$ are both even boundary data, then we say that $P_0$ is steeper than $\varphi_0$ if there exists a $\Theta_0 \in (0,\pi)$ such that $P_0(\theta) \geq \varphi_0(\theta)$ for $\theta \in [0,\Theta_0)$ while $P_0(\theta) \leq \varphi_0(\theta)$ for $\theta \in (\Theta_0,\pi]$.*

We now state our results.

**Theorem 5.2** *Suppose $\varphi_0, P_0 \in C(\partial\mathcal{D})$ are both even boundary data with $P_0$ steeper than $\varphi_0$. Also suppose that $\Phi(\varphi_0) = -1$ and that for $u_0, U_0$ the solutions to the 2MSE with boundary data $\varphi_0, P_0$ respectively:*

$$[\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0] \cap [\underline{\lim}_{r\to 0} U_0, \overline{\lim}_{r\to 0} U_0] \neq \emptyset.$$

*Then $\Phi(P_0) = -1$ as well.*

This theorem will also be used to prove Conjecture 9.2 of [R07]:

**Theorem 5.3** *Suppose $\varphi_0, \psi_0 \in C(\partial\mathcal{D})$ are both even boundary data with $\Phi(\varphi_0) = \Phi(\psi_0) \neq 0$. Then there is a continuous deformation $\varphi_{s,0}$ of even continuous functions with $\varphi_{0,0} = \varphi_0$ and $\varphi_{1,0} = \psi_0$ such that for each $s \in [0,1]$ we have $\Phi(\varphi_{s,0}) = \Phi(\varphi_0) = \Phi(\psi_0)$.*

Finally, Theorem 5.2 will be used to show boundary data which is steep on a portion of $\partial\mathcal{D}$ and relatively flat elsewhere gives a discontinuous solution to the 2MSE:

**Theorem 5.4** *For all $\vartheta_0 \in (0,\pi)$ and $M > 0$, there exists an $\epsilon > 0$ so that $\Phi(\varphi_0) = -1$ for all even boundary data $\varphi_0 \in C(\partial\mathcal{D})$ with $\|\varphi_0\|_{C(\partial\mathcal{D})} < M + \epsilon$, and $\varphi_0(\theta) \geq M$ for $\theta \in [0, \vartheta_0 - \epsilon]$ while $\varphi_0(\theta) \leq -M$ for $\theta \in [\vartheta_0 + \epsilon, \pi]$.*

# 6 Proof of Theorem 5.2

Before we prove Theorem 5.2, we prove the following lemma, which does not appear explicitly in [R07]:

**Lemma 6.1** *Suppose $\varphi_0 \in C(\partial\mathcal{D})$ is even boundary data with $\Phi(\varphi_0) = -1$ and let $u_0$ be the solution to the 2MSE with boundary data $\varphi_0$. Then for $a = \underline{\lim}_{r\to 0} u_0$ and $b = \overline{\lim}_{r\to 0} u_0$, there exists a $\rho > 0$ such that*

$$G \cap B_\rho((0,b)) \subset \{(re^{i\theta}, u(r,\theta)) : r \in (0,\rho), \theta \in (-3\pi/4, 3\pi/4)\}$$

*and*

$$G \cap B_\rho((0,a)) \subset \{(re^{i\theta}, u(r,\theta)) : r \in (0,\rho), \theta \in (2\pi - 3\pi/4, 2\pi + 3\pi/4)\}$$

**Proof:** We consider $b = \overline{\lim}_{r\to 0} u_0$. Since $\Phi(\varphi_0) = -1$, then by Theorem 4.5 we can choose $t_0 \in (\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0)$ close to $b$ so that $\theta_1(t_0) \in (\pi/2, 3\pi/4)$ and $\theta_2(t_0) \in (5\pi/4, 3\pi/2)$ where $H_j(t_0) = \{\theta = \theta_j(t_0)\}$ for $H_j(t_0)$ the tangent vertical half-plane of the surface $L_j(t_0)$, choosing $\rho_0$ sufficiently small so that by Theorem 4.4 $G \cap B_{\rho_0}((0, t_0)) = L_1(t_0) \cup L_2(t_0)$.

Hence, for each $j = 1, 2$

$$G_j \cap B_\rho((0, t_0)) \cap \{(x, t_0) : x \in \mathcal{D}_\rho\} = \mathrm{graph}_{\gamma_{j,t_0}} u_j$$

where $\gamma_{j,t_0} \subset \Omega_0$ is a continuous curve asymptotic at the origin to the ray $\{re^{i\theta_j(t_0)} : r > 0\}$. By considering $\Omega_0 \cap \mathcal{D}_{\rho_0} \setminus \gamma_{j,t_0}$, it follows that $\{re^{i\theta} \in \Omega_0 \cap \mathcal{D}_{\rho_0} : u_j(r, \theta) > t_0\}$ is a connected open set. Furthermore, since $\Phi(\varphi_0) = -1$ so that $(-1)^{j-1}\frac{\partial u_j}{\partial \theta} < 0$ for $re^{i\theta} \in \gamma_{j,t_0}$, then

$$\left(\partial\{re^{i\theta} \in \Omega_0 \cap \mathcal{D}_{\rho_0} : u_j(r, \theta) > t_0\}\right) \cap \mathcal{D}_{\rho_0} = \gamma_{j,t_0} \cup \{(x_1, 0) : x_1 \geq 0\}.$$

Since $\gamma_{j,t_0}$ is asymptotic at the origin to the ray $\{re^{i\theta_j(t_0)} : r > 0\}$ with $\theta_1(t_0) \in (\pi/2, 3\pi/4)$ and $\theta_2(t_0) \in (5\pi/4, 3\pi/2)$, then $u_1(r, \theta) = u_0(r^{1/2}, \theta/2)$ and $u_2(r, \theta) = u_0(r^{1/2}, \theta/2 + \pi)$ imply $\{(re^{i\theta}, u(r, \theta)) : r \in (0, \rho_0), u(r, \theta) > t_0\} \subseteq$

$$\{(re^{i\theta}, u(r, \theta)) : r \in (0, \rho), \theta \in (-3\pi/4, 3\pi/4)\}.$$

This gives the first part of Lemma 6.1, for $\rho \in (0, \rho_0)$ with $\rho < b - t_0$. A similar argument holds at the bottom endpoint $a$. $\Box$

We also need the following lemma for the proof of Theorem 5.2, as well as for theorems presented in §6.

**Lemma 6.2** *Suppose $\varphi_0, P_0 \in C(\partial\mathcal{D})$ are even boundary data with $\Phi(\varphi_0) = -1$ but $\Phi(P_0) \neq -1$. Then, for $u_0, U_0$ the solutions to the 2MSE with respective boundary data $\varphi_0, P_0$, there exists a $\sigma > 0$ sufficiently small so that*

$$(0, \sigma) \times \{0\} \subset \{x \in \mathcal{D} : U_0(x) - u_0(x) < A - b\}$$

*and*

$$(-\sigma, 0) \times \{0\} \subset \{x \in \mathcal{D} : U_0(x) - u_0(x) > B - a\}$$

*where $a = \underline{\lim}_{r\to 0} u_0, b = \overline{\lim}_{r\to 0} u_0, A = \underline{\lim}_{r\to 0} U_0,$ and $B = \overline{\lim}_{r\to 0} U_0$.*

Note that necessarily we have

$$\lim_{r\to 0}(U_0(r,0) - u_0(r,0)) = A - b$$

and

$$\lim_{r\to 0}(U_0(r,\pi) - u_0(r,\pi)) = B - a.$$

**Proof:** We first take the case $\Phi(P_0) = 0$. Let

$$G_u = \{(re^{i\theta}, u(r,\theta)) : r \in (0,1), \theta \in [0,4\pi)\}$$

and

$$G_U = \{(re^{i\theta}, U(r,\theta)) : r \in (0,1), \theta \in [0,4\pi)\}$$

where $u(r,\theta) = u_0(r^{1/2}, \theta/2)$ and $U_0(r,\theta) = U_0(r^{1/2}, \theta/2)$.

For $\delta > 0$ small to be chosen later, and $b = \overline{\lim}_{r\to 0} u_0$, we can choose by Theorem 4.5 $\rho > 0$ sufficiently small so that $\overline{G}_u \cap B_\rho((0,b)) =$

$$\{(w_u((0,x_2,t)), x_2, t) : (0,x_2,t) \in B_\rho((0,b))\} \cap B_\rho((0,b))$$

where $w_u$ has scaled $C^1$ norm less than $\delta$ and $w_u((0,t)) = 0$ for each $t \in (b-\rho, b]$. Using one-dimensional Calculus we therefore have

$$|w_u((0,t))| \leq \int_0^1 |Dw_u((0,(t-b)\tau + t))| \cdot |t-b| \; d\tau \leq \delta(t-b)$$

for all $t > b$.

For $t \in (b, b+\rho/2)$

$$|(w_u((0,t)), 0, t) - (0,b)| \leq \delta|t-b| + |t-b| < \rho,$$

and so $(w_u((0,t)), 0, t) \in \overline{G_u} \cap B_\rho((0,b))$ for each $t \in (b, b+\rho/2)$ (assuming $\delta < 1$). Since $b = \overline{\lim}_{r\to 0} u_0$, then we must have either $w_u((0,t)) > 0$ for all $t \in (b, b+\rho/2)$ or $w_u((0,t)) < 0$ for all $t \in (b, b+\rho/2)$. However, by Lemma 6.1 we can suppose that $\rho$ has been chosen sufficiently small so that

$$G_u \cap B_\rho((0,b)) \subset \{(re^{i\theta}, u(r,\theta)) : r \in (0,\rho), \theta \in (-3\pi/4, 3\pi/4)\}.$$

This implies that we must have $w_u((0,t)) > 0$ for all $t \in (b, b+\rho/2)$. In terms of the single-valued function $u_0$, then for $t \in (b, b+\rho/2)$

$$u_0(w_u((0,t))^{1/2}, 0) = u(w_u((0,t)), 0) = t$$

where $w_u((0,t)) \in (0, \delta(t-b))$.

Consider then $U_0(w_u((0,t))^{1/2}, 0)$ for each $t \in (b, b+\rho/2)$. Since $U_0$ can be extended continuously across the origin, then by Theorem 4.3 the component functions $U_1(r,\theta) = U_0(r^{1/2}, \theta/2)$ and $U_2(r,\theta) = U_0(r^{1/2}, \theta/2 + \pi)$ are uniformly Hölder continuous with Hölder exponent $\alpha = \alpha(1/2, \|P_0\|_{C(\partial\mathcal{D})})$ and Hölder constant $C = C(1/2, \|P_0\|_{C(\partial\mathcal{D})})$ in the region

$$\{re^{i\theta} : r \in (0, 1/2), \theta \in (\pi + 1/2, 3\pi + 1/2)\}.$$

This implies, using one-dimensional Calculus,

$$U_0(w_u((0,t))^{1/2}, 0) = U_0(w_u((0,t))^{1/2}, 2\pi) = U_2(w_u((0,t)), 2\pi)$$

$$\leq U_2(0) + \int_0^1 |DU_2(sw_u((0,t)), 2\pi)| \cdot w_u((0,t))\ ds$$

$$\leq U_0(0) + \int_0^1 \Big( |DU_2(sw_u((0,t)), 2\pi) - DU_2(0)| + |DU_2(0)| \Big) \cdot w_u((0,t))\ ds$$

$$\leq U_0(0) + C \cdot \Big( w_u((0,t)) \Big)^{1+\alpha} + |DU_2(0)| \cdot w_u((0,t))$$

since $0 < w_u((0,t)) < \delta(t-b) < 1/2$, choosing $\rho < 1$.

We thus have, for $t \in (b, b+\rho/2)$, $U_0(w_u((0,t))^{1/2}, 0) - u_0(w_u((0,t))^{1/2}, 0)$

$$\leq U_0(0) - t + C \cdot \Big( w_u((0,t)) \Big)^{1+\alpha} + |DU_2(0)| \cdot w_u((0,t))$$

$$= U_0(0) - b + \left[ C \cdot \Big( w_u((0,t)) \Big)^{1+\alpha} + |DU_2(0)| \cdot w_u((0,t)) - (t-b) \right]$$

$$< U_0(0) - b + \left[ C(\delta(t-b))^{1+\alpha} + |DU_2(0)| \cdot \delta(t-b) - (t-b) \right].$$

Choosing $\delta > 0$ initially small so that

$$C\delta^\alpha + |DU_2(0)| \cdot \delta < 1,$$

then

$$C(\delta(t-b))^{1+\alpha} + |DU_2(0)| \cdot \delta(t-b) - (t-b) < 0$$

so that

$$U_0(w_u((0,t))^{1/2}, 0) - u_0(w_u((0,t))^{1/2}, 0) < U_0(0) - b.$$

Finally, since $w_u((0,t)) \to 0$ continuously as $t \to b$, then we can set $\sigma = (w_u((0, b+\rho/2)))^{1/2}$ to get $U_0(r,0) - u_0(r,0) < U_0(0) - b$ for all $r \in (0,\sigma)$. Using Lemma 6.1 with $a = \underline{\lim}_{r\to 0} u_0$ we can similarly show that, taking $\sigma > 0$ smaller if necessary, $U_0(r,\pi) - u_0(r,\pi) > U_0(0) - a$ for all $r \in (0,\sigma)$.

Consider now the case $\Phi(P_0) = 1$. For $A = \underline{\lim}_{r\to 0} U_0$ we can also suppose that $\rho > 0$ is sufficiently small so that $\overline{G}_U \cap B_\rho((0,A)) =$

$$\{(w_U((0,x_2,t)), x_2, t) : (0,x_2,t) \in B_\rho((0,A))\} \cap B_\rho((0,A))$$

where $w_U$ has scaled $C^1$ norm less than $\delta$ and $w_U((0,t)) = 0$ for each $t \in [A, A+\rho)$. In this case we have $|w_U((0,t))| \leq \delta(A-t)$ for all $t < A$.

Since $\delta < 1$, we again have $w_U((0,t)) \in \overline{G}_U \cap B_\rho((0,A))$ for each $t \in (A-\rho/2, A)$. Recalling that $\Phi(P_0) = 1$, we can argue as in Lemma 6.1 to show that

$$G_U \cap B_\rho((0,A)) \subset \{(re^{i\theta}, U(r,\theta)) : r \in (0,\rho), \theta \in (-3\pi/4, 3\pi/4)\},$$

which implies $w_U((0,t)) > 0$ for all $t \in (A-\rho/2, A)$.

If we consider now the single-valued functions $u_0$ and $U_0$, then we have for all $t \in (0,\rho/2)$

$$u_0(w_u((0,b+t))^{1/2}, 0) = u(w_u((0,b+t)),0) = b+t > b$$

and

$$U_0(w_U((0,A-t))^{1/2}, 0) = U(w_U((0,A-t)),0) = A-t < A.$$

Since both $w_u((0,b+t)), w_U((0,A-t)) \to 0$ continuously as $t \to 0$, then there is a $\sigma > 0$ such that for $r \in (0,\sigma)$ $U_0(r,0) - u_0(r,0) < A - b$. We can similarly show that for $\sigma > 0$ sufficiently small $U_0(r,\pi) - u_0(r,\pi) > B - a$. $\square$

**Proof of Theorem 5.2:** Suppose that $\varphi_0, P_0 \in C(\partial\mathcal{D})$ are both even boundary data such that $P_0$ is steeper than $\varphi_0$, that is there is a $\Theta_0 \in (0,\pi)$ such that $P_0(\theta) \geq \varphi_0$ for $\theta \in [0,\Theta_0)$, while $P_0(\theta) \leq \varphi_0(\theta)$ for $\theta \in (\Theta_0, \pi]$. Suppose $\Phi(\varphi_0) = -1$, $[\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0] \cap [\underline{\lim}_{r\to 0} U_0, \overline{\lim}_{r\to 0} U_0] \neq \emptyset$ for $u_0, U_0$ the solutions to the 2MSE with boundary data $\varphi_0, P_0$ respectively, and for contradiction suppose $\Phi(P_0) \neq -1$.

Now, $[\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0] \cap [\underline{\lim}_{r\to 0} U_0, \overline{\lim}_{r\to 0} U_0] \neq \emptyset$ implies

$$A = \overline{\lim_{r\to 0}} U_0 \leq b = \overline{\lim_{r\to 0}} u_0$$

and

$$B = \overline{\lim_{r\to 0}} U_0 \geq a = \overline{\lim_{r\to 0}} u_0.$$

Since $\Phi(\varphi_0) = -1$ and $\Phi(P_0) \neq -1$, then let $O^-$ be the connected component of $\{x \in \mathcal{D} : U_0(x) - u_0(x) < 0\}$ containing $(0, \sigma) \times \{0\}$, and likewise let $O^+$ be the open component of $\{x \in \mathcal{D} : U_0(x) - u_0(x) > 0\}$ containing $(-\sigma, 0) \times \{0\} \subset O^+$, where $\sigma > 0$ is as in Lemma 6.2. We claim that

$$\partial O^- \cap \{e^{i\theta} : P_0(\theta) - \varphi_0(\theta) < 0\} \neq \emptyset$$

and

$$\partial O^+ \cap \{e^{i\theta} : P_0(\theta) - \varphi_0(\theta) > 0\} \neq \emptyset.$$

To show this, suppose $\partial O^- \cap \{e^{i\theta} : P_0(\theta) - \varphi_0(\theta) < 0\} = \emptyset$ for contradiction, so that

$$(\partial O^-) \setminus \{0\} \subset \{x \in \overline{\mathcal{D}} : U_0(x) - u_0(x) = 0\}.$$

The Maximum Principle of Theorem 4.2 hence implies $U_0 - u_0$ achieves its minimum on $O^-$ at the origin. However,

$$A - b \leq \underline{\lim}_{x\to 0, x\in O^-} U_0 - \overline{\lim}_{x\to 0, x\in O^-} u_0 \leq \underline{\lim}_{x\to 0, x\in O^-} (U_0 - u_0)$$

while on the other hand, by Lemma 6.2

$$U_0(r, 0) - u_0(r, 0) < A - b$$

for $r \in (0, \sigma)$ where $(0, \sigma) \times \{0\} \subset O^-$. This is a contradiction, and so $\partial O^- \cap \{e^{i\theta} : P_0(\theta) - \varphi_0(\theta) < 0\} \neq \emptyset$.

Take thus a point $e^{i\theta^-} \in \partial O^- \cap \{e^{i\theta} : P_0(\theta) - \varphi_0(\theta) < 0\}$, where $\theta^- \in (\Theta_0, 2\pi - \Theta_0)$. Since $O^-$ is connected, then we can find a continuous curve $\gamma^-$ with $\gamma^-((0, 1)) \subset O^-$, $\gamma^-(1) = e^{i\theta^-}$, and $\gamma^-(s) = (s, 0)$ for $s \in [0, \sigma)$. Also, since $u_0, U_0$ are both even in $\theta$, then we can take $\theta^- \in (\Theta_0, \pi]$ and $\gamma^- \subset \{x \in \mathcal{D} : x_2 \geq 0\}$.

We can similarly find a point $e^{i\theta^+} \in \partial O^+ \cap \{e^{i\theta} : P_0(\theta) - \varphi_0(\theta) > 0\}$ with $\theta_+ \in (-\Theta_0, \Theta_0)$, and a continuous curve $\gamma^+ \subset \{x \in \mathcal{D} : x_2 \geq 0\}$ with $\gamma^+((0, 1)) \subset O^+$, $\gamma^+(1) = e^{i\theta^+}$, and $\gamma^+(s) = (-s, 0)$ for $s \in [0, \sigma)$. This now implies that $\gamma^-$ and $\gamma^+$ must intersect, which is impossible. Hence, Theorem 5.2 must hold. $\Box$

# 7 Comparison Lemmas

Here we prove three lemmas we shall need to prove Theorems 5.3, 5.4.

**Lemma 7.1** *If $\varphi_0 \in C(\partial\mathcal{D})$ is cosine-like, then $\Phi(\varphi_0) = -1$.*

**Proof:** Suppose for contradiction that $\varphi_0$ is cosine-like but that $\Phi(\varphi_0) = 1$. Now, since $\varphi_0(\theta + \pi) = -\varphi_0(\theta)$, then by uniqueness $u_0(r, \theta + \pi) = -u_0(r, \theta)$ where $u_0$ is the solution to the 2MSE with boundary data $\varphi_0$. Since $u_0$ is also even in $\theta$, then

$$u_0(r, -\pi/2) = -u_0(r, \pi/2) = -u_0(r, -\pi/2)$$

so that $u_0(r, -\pi/2) = u_0(r, \pi/2) = 0$ for $r \in (0, 1)$.

Consider $O = \{re^{i\theta} : r \in (0, 1), \theta \in (-\pi/2, \pi/2)\}$, since $\varphi_0(\theta) > 0$ for $\theta \in (-\pi/2, \pi/2)$ then $u_0(r, \theta) \geq 0$ on $(\partial O) \setminus \{0\}$. On the other hand, since $\Phi(\varphi_0) = 1$, then by (an analogous version of the proof of) Lemma 6.2 there is a $\sigma > 0$ so that $u_0(r, 0) < a = \underline{\lim}_{r\to 0} u_0$ for all $r \in (0, \sigma)$, where $a < 0$ by the last part of Theorem 4.6. However, this contradicts the fact that, by the Maximum Principle of Theorem 4.2, $u_0$ must achieve its minimum on $O$ at $\partial O$. $\square$

**Lemma 7.2** *Suppose $\varphi_0 \in C(\partial\mathcal{D})$ is even boundary data with $\Phi(\varphi_0) = -1$. Let $u_0$ be the solution to the 2MSE with boundary data $\varphi_0$ and let $a = \underline{\lim}_{r\to 0} u_0, b = \overline{\lim}_{r\to 0} u_0$. if $P_0 \in C(\partial\mathcal{D})$ is even boundary data with $\|\varphi_0 - P_0\| < (b - a)/2$, then $\Phi(P_0) = -1$ as well.*

**Proof:** Suppose for contradiction $\Phi(P_0) \neq -1$, and let $U_0$ be the solution to the 2MSE with boundary data $P_0$. By Lemma 6.2 $\lim_{r\to 0}(U_0(r, 0) - u_0(r, 0)) = A - b$ where $A = \underline{\lim}_{r\to 0} U_0$. On the other hand, the Maximum Principle of Theorem 4.2 implies

$$-\frac{b-a}{2} < \lim_{r\to 0}(U_0(r, 0) - u_0(r, 0)) = A - b$$

so that $A > \frac{b+a}{2}$. We similarly have

$$B - a = \lim_{r\to 0}(U_0(r, \pi) - u_0(r, \pi)) < \frac{b-a}{2}$$

so that $B < \frac{b+a}{2}$ for $B = \overline{\lim}_{r\to 0} U_0$, which contradicts $A \leq B$. $\square$

The following corollary of Theorem 5.2 improves on Lemma 7.2, when $P_0$ is assumed to be steeper than $\varphi_0$.

**Corollary 7.3** *Suppose $\varphi_0, P_0 \in C(\partial\mathcal{D})$ are both even boundary data such that $P_0$ is steeper than $\varphi_0$. Also suppose $\Phi(\varphi_0) = -1$, and*

$$\|\varphi_0 - P_0\|_{C(\partial\mathcal{D})} \leq b - a$$

*for $a = \overline{\lim}_{r\to 0} u_0$ and $b = \overline{\lim}_{r\to 0} u_0$, where $u_0$ is the solution to the 2MSE with boundary data $\varphi_0$. Then $\Phi(P_0) = -1$ as well.*

**Proof:** Suppose for contradiction $\Phi(P_0) \neq -1$, and let $U_0$ be the solution to the 2MSE with boundary data $P_0$. Then the Maximum Principle of Theorem 4.2 and Lemma 6.2 imply

$$-(b-a) < \lim_{r\to 0}(U_0(r,0) - u_0(r,0)) = A - b$$

for $A = \underline{\lim}_{r\to 0} U_0$. We therefore have $a < A$, and we can similarly show $B < b$. On the other hand, this implies $[\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0] \cap [\underline{\lim}_{r\to 0} U_0, \overline{\lim}_{r\to 0} U_0] \neq \emptyset$, which contradicts Theorem 5.2 since $\Phi(P_0) \neq -1$ and $P_0$ is steeper than $\varphi_0$. $\square$

The following lemma will be used in the proof of Theorem 5.3:

**Lemma 7.4** *Suppose $\varphi_0, P_0 \in C(\partial\mathcal{D})$ are both even boundary data such that $P_0$ is steeper than $\varphi_0$, and that $P_0(\theta) - \varphi_0(\theta)$ is nonincreasing for $\theta \in [0,\pi]$ (and hence nondecreasing for $\theta \in [\pi, 2\pi]$). If $\Phi(\varphi_0) = -1$, then $\Phi(P_0) = -1$ as well.*

**Proof:** Suppose for contradiction that $\Phi(P_0) \neq -1$, then by Theorem 5.2 we must have $[\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0] \cap [\underline{\lim}_{r\to 0} U_0, \overline{\lim}_{r\to 0} U_0] = \emptyset$ where $u_0, U_0$ are the solutions to the 2MSE with boundary data $\varphi_0, P_0$ respectively. Suppose without loss of generality that

$$b = \overline{\lim_{r\to 0}}\, u_0 < A = \underline{\lim_{r\to 0}}\, U_0.$$

The Maximum Principle of Theorem 4.2 implies

$$A - b = \lim_{r\to 0}(U_0(r,0) - u_0(r,0)) < P_0(0) - \varphi_0(0).$$

Consider the boundary data $\tilde{P}_0(\theta) = P_0(\theta) - (A-b)$, then we have

$$\tilde{P}_0(0) = P_0(0) - (A-b) > \varphi_0(0),$$

$\tilde{P}_0 - \varphi_0$ is nonincreasing on $(0, \pi)$, and

$$\tilde{P}_0(\pi) = P_0(\pi) - (A - b) < \varphi_0(\pi)$$

as well since $A - b > 0$. This implies that the boundary data $\tilde{P}_0$ is steeper than the boundary data $\varphi_0$. On the other hand, the solution to the 2MSE with boundary data $\tilde{P}_0$ is $\tilde{U}_0(r, \theta) = U_0(r, \theta) - (A - b)$, so that $\Phi(\tilde{P}_0) \neq -1$. However,

$$\underline{\lim}_{r \to 0} \tilde{U}_0 = b \in [\underline{\lim}_{r \to 0} u_0, \overline{\lim}_{r \to 0} u_0],$$

contradicting Theorem 5.2. $\square$

# 8 Proof of Theorem 5.3

We begin by making two simplifications. First, it is sufficient to show that given even boundary data $\varphi_0 \in C(\partial \mathcal{D})$ with $\Phi(\varphi_0) = -1$, then there is a continuous deformation of even continuous functions $\varphi_{s,0}$ with $\Phi(\varphi_{s,0}) = -1$ for each $s \in [0, 1]$, $\varphi_{0,0} = \varphi_0$, and such that $\varphi_{1,0}$ is a cosine-like function as in Definition 4.7. This because given any two cosine-like functions $\varphi_{1,0}, \psi_{1,0} \in C(\partial \mathcal{D})$, then $P_{s,0} = \varphi_{1,0} + s(\psi_{1,0} - \varphi_{1,0})$ for each $s \in [0, 1]$ is also cosine-like, so that $\Phi(P_{s,0}) = -1$ for each $s \in [0, 1]$ by Theorem 4.6.

Second, we can assume that $\varphi_0$ is piecewise linear continuous (in $\theta$) with vertices over the points $\{e^{i\frac{\pi k}{N}}\}_{k=0}^{2N}$. This because we can take a continuous linear approximation $\varphi_{\text{lin},0}$ of $\varphi_0$ over the points $\{e^{i\frac{\pi k}{N}}\}_{k=0}^{2N}$ with $N$ sufficiently large so that

$$\|\varphi_0 - \varphi_{\text{lin},0}\|_{C(\partial \mathcal{D})} < \frac{b - a}{2}$$

where $a = \underline{\lim}_{r \to 0} u_0$ and $b = \overline{\lim}_{r \to 0} u_0$ for $u_0$ the solution to the 2MSE with boundary data $\varphi_0$. Letting $\varphi_{s,0} = \varphi_0 + s(\varphi_{\text{lin},0} - \varphi_0)$, then

$$\|\varphi_{s,0} - \varphi_{\text{lin},0}\|_{C(\partial \mathcal{D})} < s \left( \frac{b - a}{2} \right),$$

so that $\Phi(\varphi_{s,0}) = -1$ for each $s \in [0, 1]$ by Lemma 7.2.

Suppose $\varphi_0$ is piecewise linear with vertices over the points $\{e^{i\frac{\pi k}{N}}\}_{k=0}^{2N}$, let $T_k = \varphi_0(\pi k/N)$, and for $k = 0, \ldots, 2N - 1$ let $m_k$ be the slope

$$m_k = \frac{T_{k+1} - T_k}{\pi/N}.$$

Write $m = \max |m_k|$, so $m \neq 0$ necessarily. We now describe how to vary $\varphi_0$ continuously to cosine-like boundary data.

First, choose $T^0$ so that
$$\frac{T_1 - T^0}{\pi/N} = -m,$$
so $T^0 \geq T_0$ necessarily. Let $P^0_{s,0} \in C(\partial\mathcal{D})$ be the even piecewise linear function with vertices over the points $\{e^{i\frac{\pi k}{N}}\}_{k=0}^{2N}$ and values
$$P^0_{s,0}(\pi k/N) = T_k$$
for $k = 1, \dots, 2N-1$ while
$$P^0_{s,0}(0) = T_0 + s(T^0 - T_0) = P^0_{s,0}(2\pi).$$
Let $T^0_{s,0} := P^0_{s,0}(\pi k/N)$ for $k = 0, \dots, 2N$.

Since $T^0 \geq T_0$, then $P^0_{s,0}$ is steeper than $\varphi_0$ for each $s \in [0,1]$ with $\Theta_0 = \frac{\pi}{N}$, and $P^0_{s,0}(\theta) - \varphi_0(\theta)$ is nonincreasing for $\theta \in [0,\pi]$. Lemma 7.4 implies $\Phi(P^0_{s,0}) = -1$ for each $s \in [0,1]$.

We then proceed iteratively to "raise" the boundary data $\varphi_0$ at the vertices $e^{i\frac{\pi}{N}}, \dots, e^{i\frac{\pi(N-2)}{N}}$ until the slope of each linear piece is $-m$. Precisely, suppose for some $K \in \{0, \dots N-3\}$ we have shown there is an even piecewise linear function $P^K_0 \in C(\partial\mathcal{D})$ with vertices over the points $\{e^{i\frac{\pi k}{N}}\}_{k=0}^{2N}$, and that for the slopes
$$m^K_k = \frac{T^K_{k+1} - T^K_k}{\pi/N}$$
for $k = 0, \dots, 2N-1$ where
$$T^K_k = P^K_0(\pi k/N)$$
for $k = 0, \dots, 2N$, we have $|m^K_k| \leq m$ for all $k$ and $m^K_0 = \ldots = m^K_K = -m$. Suppose also there is a continuous deformation $P^K_{s,0} \in C(\partial\mathcal{D})$ of even functions $P^K_{s,0}$ with $P^K_{0,0} = \varphi_0$, $P^K_{1,0} = P^K_0$ and $\Phi(P^K_{s,0}) = -1$ for each $s \in [0,1]$ (we have shown the case $K = 0$ above).

Then for $s \in [0,1]$, define $P^{K+1}_{s,0} \in C(\partial\mathcal{D})$ to be the even piecewise linear function with vertices at the points $\{e^{i\frac{\pi k}{N}}\}_{k=0}^{2N}$ and values
$$P^{K+1}_{s,0}(\pi k/N) = P^K_0(\pi k/N)$$

for $k = K+2, \ldots, N$. Choose on the other hand $T^{K+1}$ so that

$$\frac{P_0^{K+1}(\pi(K+2)/N) - T^{K+1}}{\pi/N} = \frac{P_0^{K}(\pi(K+2)/N) - T^{K+1}}{\pi/N} = -m,$$

necessarily $T^{K+1} \geq P^K(\pi(K+1)/N)$. For $k = 0, \ldots, K+1$ define

$$P_{s,0}^{K+1}(\pi k/N) = P_0^K(\pi k/N) + s(T^{K+1} - P_0^K(\pi(K+1)/N)).$$

Since $T_{K+1} \geq P_0^K(\pi(K+1)/N)$, then each $P_{s,0}^{K+1}$ is steeper that $P_0^K$ with $\Theta_0 = \frac{\pi(K+2)}{N}$ (note that $\Theta_0 \in (0, \pi)$ since $K \in \{0, \ldots, N-3\}$), and $P_{s,0}^{K+1}(\theta) - P_0^K(\theta)$ is nonincreasing for $\theta \in (0, \pi)$. Lemma 7.4 hence implies $\Phi(P_{s,0}^{K+1}) = -1$ for each $s \in [0,1]$.

We can thus conclude there is an even piecewise linear function $P_0^{N-2} \in C(\partial\mathcal{D})$ with vertices at $0, \frac{\pi(N-1)}{N}, \pi, \frac{\pi(2N-1)}{N}$ such that

$$\frac{P_0^{N-2}(\pi(N-1)/N) - P_0^{N-2}(0)}{\pi(N-1)/N} = -m$$

and

$$\left|\frac{P_0^{N-2}(\pi) - P_0^{N-2}(\pi(N-1)N)}{\pi/N}\right| \leq m.$$

Furthermore, there is a continuous deformation of even functions $P_{s,0}^{N-2} \in C(\partial\mathcal{D})$ with $P_{0,0}^{N-2} = \varphi_0$, $P_{1,0}^{N-2} = P_0^{N-2}$, and $\Phi(P_{s,0}^{N-2}) = -1$ for each $s \in [0,1]$.

To define $P^{N-1}$, choose in this case $T^N$ so that

$$\frac{T^N - P_0^{N-2}(\pi(N-1)/N)}{\pi/N} = -m,$$

hence $T^N \leq P_0^{N-1}(\pi)$ necessarily. For $s \in [0,1]$, we now define $P_{s,0}^{N-1} \in C(\partial\mathcal{D})$ to be the even piecewise linear function with vertices at $0, \frac{\pi(N-1)}{N}, \pi, \frac{\pi(2N-1)}{N}$ so that $P_{s,0}^{N-1}(0) = P_0^{N-2}(0)$, $P_{s,0}^{N-1}(\pi(N-1)/N) = P_0^{N-2}(\pi(N-1)/N)$, and

$$P_{s,0}^{N-1}(\pi) = P_0^{N-2}(\pi) - s(P_0^{N-2}(\pi) - T^N).$$

It follows that each $P_{s,0}^{N-1}$ is steeper than $P_0^{N-2}$ with $\Theta_0 = \frac{\pi(N-1)}{N}$, and $P_{s,0}^{N-1} - P_0^{N-2}$ is nonincreasing for $\theta \in (0, \pi)$. Lemma 7.4 implies $\Phi(P_{s,0}^N) = -1$ for each $s \in [0,1]$.

Setting $P_0 = P_{1,0}^{N-1}$, then $P_0 \in C(\partial\mathcal{D})$ is an even function which on $(0,\pi)$ is just a line with slope $-m$. To conclude the proof of Theorem 5.3, we consider the deformation $P_{s,0}(\theta) = P_0(\theta) - sP_0(\pi/2)$, then $\Phi(P_{s,0}) = -1$ for each $s \in [0,1]$ and $P_{1,0}$ is cosine-like. $\Box$

# 9 Proof of Theorem 5.4

Suppose for contradiction there exists $\vartheta_0 \in (0,\pi)$ and $M > 0$ with a sequence of $\epsilon_k \to 0$ and even boundary data $\varphi_{k,0} \in C(\partial\mathcal{D})$ such that $\|\varphi_{k,0}\|_{C(\partial\mathcal{D})} < M + \epsilon_k$, $\varphi_{k,0}(\theta) \geq M$ for $\theta \in [0, \vartheta_0 - \epsilon_k]$, $\varphi_{k,0}(\theta) \leq -M$ for $\theta \in [\vartheta_0 + \epsilon_k, \pi]$, and the solution $u_{k,0}$ to the 2MSE with boundary data $\varphi_{k,0}$ can be extended continuously across the origin. By Theorem 4.3 we have

$$\|u_{k,0}\|_{C^1(\mathcal{D}_\sigma)} \leq C(\sigma, M + \epsilon_k)$$

for each $\sigma < 1$. Hence by compactness, we can assume that the functions $u_{k,0}$ converge uniformly on compact subsets of $\mathcal{D}$ to $U_0 \in C^\infty(\mathcal{D})$ an even solution to the 2MSE in $\mathcal{D} \setminus \{0\}$.

Furthermore, each two-valued function $u_k(r,\theta) = u_{k,0}(r^{1/2}, \theta/2)$ is locally in $r, \theta$ a solution to the MSE, with continuous boundary data

$$\begin{gathered} M \leq \varphi_{k,0}(\theta/2) < M + \epsilon_k \text{ for } \theta \in [-2\vartheta_0 + 2\epsilon_k, 2\vartheta_0 - 2\epsilon_k] \\ -M - \epsilon_k \leq \varphi_{k,0}(\theta/2) < -M \text{ for } \theta \in [2\vartheta_0 + 2\epsilon_k, 4\pi - 2\vartheta_0 - 2\epsilon_k]. \end{gathered}$$

Standard local barriers for solutions to the MSE then give us uniform continuity estimates for the functions $u_k(r,\theta)$ near each point $e^{i\theta}$ with $\theta \in (-2\vartheta_0, 2\vartheta_0) \cup (2\vartheta_0, 4\pi - 2\vartheta_0)$. Hence, letting $P_0$ be the step function

$$\begin{gathered} P_0(\theta) = M \text{ for } \theta \in (-\vartheta_0, \vartheta_0) \\ P_0(\theta) = -M \text{ for } \theta \in (\vartheta_0, 2\pi - \vartheta_0), \end{gathered}$$

then $U_0|_{\partial\mathcal{D}\setminus\{e^{i\vartheta_0}, e^{-i\vartheta_0}\}} = P_0$ continuously.

The Maximum Principle of Theorem 4.2 implies $\|u_{k,0}\|_{C(\mathcal{D})} \leq M + \epsilon_k$ for each $k$, and so $\|U_0\|_{C(\mathcal{D})} \leq M$. Since $U_0$ cannot attain its maximum or minimum in $\mathcal{D}$, then $-M < U_0(0) < M$.

Consider now the two-valued graph $G$ corresponding to $U_0$

$$G = \{(re^{i\theta}, U(r,\theta)) : r \in (0,1), \theta \in [0, 4\pi)\}$$

where $U(r,\theta) = U_0(r^{1/2}, \theta/2)$. Consider $\Omega_0 = \{re^{i\theta} : r \in (0,1), \theta \in (0, 2\pi)\}$, and consider the component functions $U_1(r,\theta) = U_0(r^{1/2}, \theta/2)$ and $U_2(r,\theta) = U_0(r^{1/2}, \theta/2 + \pi)$ over $\Omega_0$. Observe that $U_1(1,\theta) = M$ for $\theta \in (0, 2\vartheta_0)$ and $U_1(1,\theta) = -M$ for $\theta \in (2\vartheta_0, 2\pi)$ continuously, while $U_2(1,\theta) = -M$ for $\theta \in (0, 2\pi - 2\vartheta_0)$ and $U_2(1,\theta) = M$ for $\theta \in (2\pi - 2\vartheta_0, 2\pi)$ continuously.

Consider the graph of the first component function $G_1 = \operatorname{graph}_{\Omega_0} U_1$. Continuity of $U_0$ in $\mathcal{D}$, along with $U_0|_{\partial\mathcal{D}\setminus\{e^{i\vartheta_0}, e^{-i\vartheta_0}\}} = P_0$ continuously, implies that $\partial G_1 \setminus \{(x_1, 0, t) : x_1 \in [0,1], t \in \mathbb{R}\}$ is the union of the arcs

$$\{(e^{i\theta}, M) : \theta \in (0, 2\vartheta_0)\}, \{(e^{i\theta}, -M) : \theta \in (2\vartheta_0, 2\pi - 2\vartheta_0)\}$$

and the vertical line segment $\{e^{i2\vartheta_0}\} \times [-M, M]$.

We claim that since $\partial G_1$ contains the vertical line segment $\{e^{i2\vartheta_0}\} \times [-M, M]$ we can apply the arguments proving Theorem 3 of [SW07] and Theorem 4.2 of [R07] to show that for each $t \in (-M, M)$ there is a $\rho > 0$ sufficiently small so that

$$G_1 \cap B_\rho((e^{i2\vartheta_0}, t)) = \mathcal{L}_1(t)$$

where $\mathcal{L}_1(t)$ is a smooth embedded minimal surface-with-boundary $\{e^{i2\vartheta_0}\} \times (t - \rho, t + \rho)$. We highlight the important steps of the argument, noting that they rely heavily on the Theory of Varifolds and the Theory of Embedded Stable Minimal Surfaces (we refer the reader to [S83] and [SS81]).

First, let $R_{2\vartheta_0}$ be the reflection across the axis $\{e^{i2\vartheta_0}\} \times \mathbb{R}$ given by $R_{2\vartheta_0}(x,t) = (2e^{i2\vartheta_0} - x, t)$. The Reflection Principle of Allard [A75] implies the two-dimensional varifold

$$G_1 \cup R_{2\vartheta_0,\#} G_1$$

is stationary in $B_\rho((e^{i2\vartheta_0}, t))$ for each $t \in (-M, M)$ and $\rho < \min\{M - t, M + t\}$. This means that the Monotonicity Formula (see [A72]) holds for $G_1 \cup R_{2\vartheta_0,\#} G_1$ and hence $G_1$ at each point $(e^{i2\vartheta_0}, t)$ with $t \in (-M, M)$. So the density of $G_1$ exists with

$$\Theta_{G_1}((e^{i2\vartheta_0}, t)) \geq 1/2$$

for each $t \in (-M, M)$.

From this we can consider points of almost local minimum density as in the proof of Theorem 3 in [SW07]. For any subinterval $[a,b] \subset (-M,M)$, we let

$$\kappa = \inf_{t\in[a,b]} \Theta_{G_1}((e^{i2\vartheta_0},t)).$$

Using the varifold compactness argument of Lemma A, [SW07], we can show that for any $\delta > 0$, there is a $t_0 \in [a,b]$ and a $\rho_0 > 0$ so that for any $t \in (t_0-\rho, t_0+\rho)$ and $\rho \in (0,\rho_0)$, there are half-planes $\mathcal{H}_1(t,\rho),\ldots,\mathcal{H}_{p(t,\rho)}(t,\rho)$ with $\partial\mathcal{H}_j(t,\rho) = \{e^{i2\vartheta_0}\} \times \mathbb{R}$ such that

$$\text{Hausdorff distance}(G_1 \cap B_\rho((e^{i2\vartheta_0},t)), \bigcup_{j=1}^{p(t,\rho)} \mathcal{H}_j) < \delta\rho$$

where $p(t,\rho) \in \{1,\ldots,p_0\}$ for some $p_0 \in \mathbb{N}$ depending on $\Theta_{G_1}((e^{i2\vartheta_0},t_0))$.

Using the regularity theory of stable embedded minimal surfaces from [SS81], and arguing as in the proof of Theorem 3 of [SW07], we can choose $\delta > 0$ sufficiently small, and then $t_0 \in [a,b]$ and $\rho_0 > 0$ corresponding so that

$$G_1 \cap B_{\rho_0}((e^{i2\vartheta_0},t_0)) = \bigcup_{j=1}^{q_0(t_0)} \mathcal{L}_j(t_0)$$

where each $\mathcal{L}_j(t_0)$ is a smooth embedded minimal surface-with-boundary $\{e^{i2\vartheta_0}\} \times (t_0-\rho_0, t_0+\rho_0)$, and

$$q_0(t_0) = 2\Theta_{G_1}((e^{i2\vartheta_0},t_0)).$$

Second, we use Lemma B of [SW07] to conclude $q_0(t_0) = 1$. For this, note

$$\mathcal{L}_j(t_0) \subset \{(x,t) \in \mathcal{D} \times \mathbb{R} : (x - e^{i2\vartheta_0}) \cdot e^{i2\vartheta_0} < 0\}.$$

Hence, if $q_0(t_0) > 1$, then we could find an open region $\Omega \subset \{x \in \mathcal{D} : |x - e^{i2\vartheta_0}| < \rho_0\}$, contained in the open half-space $\{x : (x - e^{i2\vartheta_0}) \cdot e^{i2\vartheta_0} < 0\}$, with $e^{i2\vartheta_0} \in \partial\Omega$,

$$U_1|_{\partial\Omega\setminus\{e^{i2\vartheta_0}\}} = t_0$$

continuously, and so that $U_1$ is a solution to the MSE in $\Omega$ which cannot be extended continuously at $e^{i2\vartheta_0}$. This directly contradicts Lemma B of [SW07], which uses a barrier argument to show that for an open region $\Omega$ contained in an open half-space with $\partial\Omega$ intersecting the half-space at a single point $x_0$, then any solution to the MSE in the region $\Omega$ which attains continuous boundary values on $\Omega \setminus \{x_0\}$ can also be extended continuously at $x_0$, if the boundary data can be extended continuously at $x_0$.

We have thus shown that there is a dense set of heights $t \in (-M, M)$, which we call "good heights," so that $G_1 \cap B_\rho((e^{i2\vartheta_0}, t)) = \mathcal{L}_1(t)$ where $\mathcal{L}_1(t)$ is a smooth embedded minimal surface-with-boundary $\{e^{i2\vartheta_0}\} \times (t-\rho, t+\rho)$. We now sketch how we can use the arguments of the proof of Theorem 4.2 of [R07] to show that every height $t \in (-M, M)$ is a good height.

First, choose any $t_0 \in (-M, M)$ and pick any good heights $t_1, t_2 \in (-M, M)$ with $t_1 < t_0 < t_2$. Choose $\rho > 0$ sufficiently small so that

$$G_1 \cap B_\rho((e^{i2\vartheta_0}, t_k)) = \mathcal{L}_1(t_k)$$

for each $k = 1, 2$ where each $\mathcal{L}_1(t_k)$ is a minimal surface-with-boundary $(t_k - \rho, t_k + \rho)$. Since $U_1$ has boundary values $U_1(1, \theta) = M$ for $\theta \in (0, 2\vartheta_0)$ and $U_1(1, \theta) = -M$ for $\theta \in (2\vartheta_0, 2\pi)$ continuously, then we can show that each surface $\mathcal{L}_1(t_k)$ is decreasing, meaning that $\frac{\partial U_1}{\partial \theta} < 0$ for each $re^{i\theta} \in \operatorname{proj}_{\Omega_0} \mathcal{L}_1(t_k)$. We can then argue as in the proof of Lemma 6.1 (see more analogously the proof of Lemma 5.1 of [R07]) to show that

$$G_1 \cap (\{x \in \mathcal{D} : |x - e^{i2\vartheta_0}| < \rho\} \times (t_1, t_2)) = \operatorname{graph}_{\Omega_1} U_1$$

where $\Omega_1 \subset \Omega_0$, and $e^{i2\vartheta_0} \in \partial\Omega_1$. Furthermore, $\Omega_1$ is contained in the wedge-shaped region $W_{t_1,t_2}$ of the half-space $\{(x,t) \in \mathbb{R}^3 : (x - e^{i2\vartheta_0}) \cdot e^{i2\vartheta_0} < 0\}$ bounded by the tangent vertical half-planes $\mathcal{H}_1(t_1), \mathcal{H}_1(t_2)$ of $\mathcal{L}_1(t_1), \mathcal{L}_1(t_2)$ at $(e^{i2\vartheta_0}, t_1), (e^{i2\vartheta_0}, t_2)$ respectively (observe that by uniqueness of the Cauchy Problem, the half-planes $\mathcal{H}_1(t_1), \mathcal{H}_1(t_2)$ must meet with angle less than $\pi$ in the half-space $\{(x,t) \in \mathbb{R}^3 : (x - e^{i2\vartheta_0}) \cdot e^{i2\vartheta_0} < 0\}$).

Second, let $G_{\Omega_1,1} = \operatorname{graph}_{\Omega_1} U_1$, then as in Lemma 5.2 of [R07] we have for every $\sigma > 0$ with $\sigma < \min\{t_0 - t_1 - \rho, t_2 - \rho - t_0\}$ that the only boundary of $G_1 \cap B_\sigma((e^{i2\vartheta_0}, t_0)) = G_{\Omega_1,1} \cap B_\sigma((e^{i2\vartheta_0}, t_0))$ in $B_\sigma((e^{i2\vartheta_0}, t_0))$ is $\{e^{i2\vartheta_0}\} \times (t_0 - \sigma, t_0 + \sigma)$. If we consider any tangent cone $\mathbb{C}$ of $G_1$ at $(e^{i2\vartheta_0}, t_0)$, then the theory of stable embedded minimal surfaces from [SS81] and the structure of one-dimensional stationary varifolds of $S^2$ from [AA76] imply that $\mathbb{C}$ must be either a nonvertical plane or a union of vertical half-planes with boundary $\{e^{i2\vartheta_0}\} \times \mathbb{R}$. Since $\mathbb{C}$ must be supported in the wedge $W_{t_1,t_2}$, then

$$\mathbb{C} = \sum_{k=1}^{q_0} \mathbb{H}_k$$

where each $\mathbb{H}_k \subset W_{t_1,t_2}$ is a vertical half-plane with boundary $\{e^{i2\vartheta_0}\} \times \mathbb{R}$.

Third, for $\delta > 0$ sufficiently small as in [SS81], we can take $\sigma > 0$ sufficiently small so that

$$(1)\quad G_1 \cap (\{x : \delta\sigma < |x - e^{i2\vartheta_0}| < \sigma\} \times (t_0 - \sigma, t_0 + \sigma)) = \bigcup_{k=1}^{q_0} \mathbb{L}_k$$

where each $\mathbb{L}_k$ is a graph over $\mathbb{H}_k$ with scaled $C^1$ norm less than $\delta$. Each surface $\mathbb{L}_k$ can also be characterized as being increasing or decreasing as we move counterclockwise in $\theta$. This together with the fact that $U_1$ achieves continuously the boundary values $U_1(1,\theta) = M$ for $\theta \in (0, 2\vartheta_0)$ and $U_1(1,\theta) = -M$ for $\theta \in (2\vartheta_0, 2\pi)$, implies that $q_0$ must be odd.

Fourth, suppose $q_0 > 1$ for contradiction, and without loss of generality since the set of good heights is an open set, that each $t \in (t_0 - \sigma, t_0)$ is a good height. By considering the cross sections $G_1 \cap \{(x,t) : x \in \mathcal{D}\}$ at heights $t \in (t_0 - \sigma, t_0)$, and observing that the half-planes $\mathbb{H}_k \subset W_{t_1,t_2}$ must meet with angle less than $\pi$ in $\{(x,t) \in \mathbb{R}^3 : (x - e^{i2\vartheta_0}) \cdot e^{i2\vartheta_0} < 0\}$, then we can show (as in the proof of Lemma 5.3 of [R07]) that there is a simply connected region

$$G^*_{\Omega_1,1} \subset G_{\Omega_1,1} \cap (\{x \in \mathcal{D} : |x - e^{i2\vartheta_0}| < \sigma\} \times (t_0 - 7\sigma/8, t_0 - \sigma/8))$$

such that

$$c(\mathbb{C})\sigma \leq \int_{G^*_{\Omega_1,1} \cap (\{x \in \mathcal{D} : |x - e^{i2\vartheta_0}| < \sigma\} \times (t_0 - 5\sigma/8, t_0 - 3\sigma/8))} |A_{G_1}| \; d\mathcal{H}^2$$

where $c(\mathbb{C}) > 0$ depends only on the tangent cone $\mathbb{C}$ (and not $\sigma$) and $A_{G_1}$ is the second fundamental form of $G_1$ (we may need to initially assume that (1) holds with $\delta$ replaced by a fixed factor depending on $q_0$ times $\delta$, in order to apply the iterative procedure of the proof of Lemma 5.5 in [R07]).

Fifth, using the Cauchy-Schwartz Inequality and the Monotonicity Formula for $G_1$ at points $(e^{i2\vartheta_0}, t)$ with $t \in (t_0 - \sigma, t_0)$, we can show as in the proof of Lemma 5.3 in [R07] that

$$\frac{c(\mathbb{C}, t_0)}{\delta} \leq \int_{G^*_{\Omega_1} \cap (\{x \in \mathcal{D} : |x - e^{i2\vartheta_0}| < \sigma\} \times (t_0 - 5\sigma/8, t_0 - 3\sigma/8))} |A_{G_1}|^2 \; d\mathcal{H}^2.$$

As opposed to the proof of Lemma 5.3 in [R07], here the constant $c(\mathbb{C}, t_0)$ depends on $t_0$. This because we do not have a uniform mass bound as in

(3.6) of [R07] for balls centered at points $(e^{i2\vartheta_0}, t)$. Nonetheless, using the Reflection Principle of [AA76], the Monotonicity Formula (see [A72] and [S83]) for $G_1$ at points on the vertical axis $\{e^{i2\vartheta_0}\} \times (-M, M)$ implies

$$\mathcal{H}^2(\{x \in \mathcal{D} : |x - e^{i2\vartheta_0}| < \sigma\} \times (t_0 - 5\sigma/8, t_0 - 3\sigma/8))) \leq c(t_0)\sigma$$

which we use to show the above inequality.

Sixth, since $G_{\Omega_1^*,1}$ is simply connected we can apply inequality (3.11) of [SS83] as in the proof of Lemma 5.3 in [R07] to conclude

$$\frac{c(\mathbb{C}, t_0)}{\delta} \leq c_{13} \left( \frac{\int_{G_1 \cap B_{\frac{6\sigma}{16}}((x, t_0 - \frac{\sigma}{2}))} d\mathcal{H}^2}{(\frac{6\sigma}{16})^2} + 1 \right)$$

where $c_{13}$ is as in [SS83], $x \in \mathcal{D}$ is some point with $|x - e^{i2\vartheta_0}| < \delta\sigma$, and

$$(x, t_0 - \sigma/2) \in G^*_{\Omega_1,1} \cap (\{x \in \mathcal{D} : |x - e^{i2\vartheta_0}| < \sigma\} \times (t_0 - 5\sigma/8, t_0 - 3\sigma/8)).$$

Seventh, since

$$B_{\frac{6\sigma}{16}}((x, t_0 - \sigma/2)) \subset B_\sigma((e^{i2\vartheta_0}, t_0))$$

for $\delta < 1/8$, then the Monotonicity Formula on $G_1$ at $(e^{i2\vartheta_0} t_0)$ implies

$$\frac{c(\mathbb{C}, t_0)}{\delta} \leq c_{13} c(t_0)$$

for some constant $c(t_0)$ depending only on $t_0$. By choosing $\delta > 0$ small initially depending on $t_0$ and $\sigma > 0$ sufficiently small corresponding, we thus have a contradiction. We have shown that every $t \in (-M, M)$ is a good height.

Proceeding with the proof of Theorem 5.4, recall that by the Maximum Principle of Theorem 4.2, $-M < U_0(0) < M$. Choose then $\epsilon > 0$ so that $-M + \epsilon < U_0(0) < M - \epsilon$, and take $\rho > 0$ sufficiently small so that

$$G_j \cap \Big( \{re^{i\theta} : r \in (1 - \rho, 1), \theta \in (2\vartheta_0 - \rho, 2\vartheta_0 + \rho)\} \times (-M + \epsilon, M - \epsilon) \Big)$$

is a smooth embedded minimal surface-with-boundary $\mathbf{L}_j$ with boundary $\{e^{i2\vartheta_0}\} \times (-M + \epsilon, M - \epsilon)$. Since $U_1|_{\partial\mathcal{D}} = P_0$, then $\mathbf{L}_1$ is decreasing, meaning that $\frac{\partial U_1}{\partial \theta} < 0$ for all $re^{i\theta} \in \mathrm{proj}_{\Omega_0} \mathbf{L}_1$. This implies, for $\rho$ sufficiently small, that for $r \in (1 - \rho, 1)$ there are angles $\theta_r^-, \theta_r^+ \in (0, \pi)$ with

$$\vartheta_0 - \rho/2 < \theta_r^- < \vartheta_0 < \theta_r^+ < \vartheta_0 + \rho/2$$

such that

$$U_1(r,\theta) > M - \epsilon \text{ for } \theta \in (2\vartheta_0 - \rho, 2\theta_r^-),$$

$$\frac{\partial U_1}{\partial \theta} < -\frac{1}{2} \text{ for } \theta \in (2\theta_r^-, 2\theta_r^+),$$

and

$$U_1(r,\theta) < -M + \epsilon \text{ for } \theta \in (2\theta_r^+, 2\vartheta_0 + \rho).$$

On the other hand, $U_0$ attains its boundary values continuously uniformly on $\overline{D}$ away from the set

$$\left\{ re^{i\theta} : r \in (1-\rho, 1], \theta \in \left(\vartheta_0 - \frac{\rho}{2}, \vartheta_0 + \frac{\rho}{2}\right) \cup \left(2\pi - \vartheta_0 - \frac{\rho}{2}, 2\pi - \vartheta_0 + \frac{\rho}{2}\right) \right\}.$$

We can therefore choose $\sigma \in (\sqrt{1-\rho}, 1)$ with $\sigma$ sufficiently close to 1 so that

$$U_0(\sigma,\theta) > M - \epsilon \text{ for } \theta \in [0, \theta_{\sigma^2}^-),$$

$$\frac{\partial}{\partial \theta} U_0(\sigma,\theta) < -1 \text{ for } \theta \in (\theta_{\sigma^2}^-, \theta_{\sigma^2}^+),$$

$$U_0(\sigma,\theta) < -M + \epsilon \text{ for } \theta \in (\theta_{\sigma^2}^+, 2\pi - \theta_{\sigma^2}^+),$$

$$\frac{\partial}{\partial \theta} U_0(\sigma,\theta) > 1 \text{ for } \theta \in (2\pi - \theta_{\sigma^2}^+, 2\pi - \theta_{\sigma^2}^-),$$

and

$$U_0(r,\theta) > M + \epsilon \text{ for } \theta \in (2\pi - \theta_{\sigma^2}^-, 2\pi),$$

since $U_1(r,\theta) = U_0(r^{1/2}, \theta/2)$ and $U_0$ is even in $\theta$.

Define the even boundary data $P_{\sigma,0} \in C(\partial\mathcal{D})$ by $P_{\sigma,0}(\theta) = \frac{1}{\sigma} U_0(\sigma,\theta)$. Then the solution to the 2MSE with boundary data $P_{\sigma,0}$ is $\frac{1}{\sigma} U_0(\sigma x)$. Since $U_0$ can be extended continuously across the origin, then $\Phi(P_{\sigma,0}) = 0$. On the other hand, we can choose $\alpha > 0$ sufficiently small so that for

$$\varphi_{\alpha,0}(\theta) = \alpha \cos(\theta) + \frac{1}{\sigma} U_0(0)$$

there is a $\Theta_0 \in (\theta_{\sigma^2}^-, \theta_{\sigma^2}^+) \subset (0,\pi)$ such that $P_{\sigma,0}(\theta) > \varphi_{\alpha,0}(\theta)$ for $\theta \in [0, \Theta_0)$, and $P_{\sigma,0}(\theta) < \varphi_{\alpha,0}(\theta)$ for $\theta \in (\Theta_0, \pi]$, meaning that $P_{\sigma,0}$ is steeper than $\varphi_{\alpha,0}$. We now have a contradiction, because $\Phi(\varphi_{\alpha,0}) = -1$ by Theorem 4.6, and for $u_{\alpha,0}$ the solution to the 2MSE with boundary data $\varphi_{\alpha,0}$

$$U_0(0) \in (\varliminf_{r\to 0} u_{\alpha,0}, \varlimsup_{r\to 0} u_{\alpha,0}),$$

while $\Phi(P_{\sigma,0}) = 0$. $\square$

# 10 Subset Comparison

First, we show by example that there are even boundary data $\varphi_0, P_0 \in C(\partial\mathcal{D})$ with $P_0$ steeper than $\varphi_0$, but so that $[\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0] \cap [\underline{\lim}_{r\to 0} U_0, \overline{\lim}_{r\to 0} U_0] \neq \emptyset$ with

$$[\varliminf_{r\to 0} u_0, \varlimsup_{r\to 0} u_0] \not\subseteq [\varliminf_{r\to 0} U_0, \varlimsup_{r\to 0} U_0]$$

for $u_0, U_0$ the solutions to the 2MSE with boundary data $\varphi_0, P_0$ respectively. For this, consider $P_0(\theta) = \cos\theta$ and

$$\varphi_0(\theta) = \alpha\cos(\theta) + \varliminf_{r\to 0} U_0$$

with $\alpha > 0$. By Theorem 4.6 (and Lemma 7.1), $\Phi(P_0) = \Phi(\varphi_0) = -1$. Let $u_0, U_0$ be the solutions to the 2MSE with boundary data $\varphi_0, P_0$ respectively.

On the other hand, the Maximum Principle of Theorem 4.2 implies $\underline{\lim}_{r\to 0} U_0 \in (-1, 1)$. Hence, we can choose $\alpha > 0$ sufficiently small so that $P_0$ is steeper $\varphi_0$. On the other hand, by the last part of Theorem 4.6, $\underline{\lim}_{r\to 0} U_0$ is the midpoint of $[\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0]$. We thus have $[\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0] \cap [\underline{\lim}_{r\to 0} U_0, \overline{\lim}_{r\to 0} U_0] \neq \emptyset$ and $[\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0] \not\subseteq [\underline{\lim}_{r\to 0} U_0, \overline{\lim}_{r\to 0} U_0]$ as claimed.

Second, we show that there are even boundary data $\varphi_0, P_0 \in C(\partial\mathcal{D})$ with $P_0$ steeper than $\varphi_0$ but so that

$$[\varliminf_{r\to 0} u_0, \varlimsup_{r\to 0} u_0] \cap [\varliminf_{r\to 0} U_0, \varlimsup_{r\to 0} U_0] = \emptyset$$

for $u_0, U_0$ the solutions to the 2MSE with boundary data $\varphi_0, P_0$ respectively. This is done by taking $P_0(\theta) = \cos\theta$ and

$$\varphi_0(\theta) = \alpha\cos(\theta) + t$$

for $t \in (-1, \liminf_{r\to 0} U_0)$ and $\alpha > 0$ with $\alpha < \min\{\underline{\lim}_{r\to 0} U_0 - t, t + 1\}$.

The above two examples notwithstanding, we do have the following:

**Theorem 10.1** *Suppose that $\varphi_0, P_0 \in C(\partial\mathcal{D})$ are (distinct) even boundary data such that $P_0$ is steeper than $\varphi_0$, and $\Phi(\varphi_0) = \Phi(P_0) = -1$. Then*

$$[\varliminf_{r\to 0} U_0, \varlimsup_{r\to 0} U_0] \not\subseteq [\varliminf_{r\to 0} u_0, \varlimsup_{r\to 0} u_0]$$

*where $u_0, U_0$ are the solutions to the 2MSE with boundary data $\varphi_0, P_0$ respectively.*

**Proof:** Suppose $[\underline{\lim}_{r\to 0} U_0, \overline{\lim}_{r\to 0} U_0] \subseteq [\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0]$ for contradiction. We proceed by showing a version of Lemma 6.2 in the case $\Phi(\varphi_0) = \Phi(P_0) = -1$. Let

$$a = \underline{\lim}_{r\to 0} u_0, b = \overline{\lim}_{r\to 0} u_0, \text{ and } A = \underline{\lim}_{r\to 0} U_0, B = \overline{\lim}_{r\to 0} U_0,$$

so that $a \leq A < B \leq b$.

Consider $\underline{\lim}_{r\to 0}(U_0 - u_0)$, then Lemma 6.1 implies

$$\underline{\lim}_{r\to 0}(U_0 - u_0) \leq \lim_{r\to 0}(U_0(r,0) - u_0(r,0)) = B - b \leq 0$$

since $\Phi(\varphi_0) = \Phi(P_0) = -1$. Noting that $a \leq A$, then we can choose $t_u \in (\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0]$ and $t_U \in [\underline{\lim}_{r\to 0} U_0, \overline{\lim}_{r\to 0} U_0]$ with $t_U \leq t_u$ so that

$$t_U - t_u = \underline{\lim}_{r\to 0}(U_0 - u_0).$$

To proceed without having to consider separate cases (such as $t_u \in (\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0)$ or $t_u = b$), we note that Theorems 4.4,4.5 together imply the following lemma:

**Lemma 10.2** *Suppose $\varphi_0 \in C(\partial\mathcal{D})$ is even boundary data with $\Phi(\varphi_0) \neq 0$, and let $u_0$ be the solution to the 2MSE with boundary data $\varphi_0$. For any $t_0 \in [\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0]$ there is a vertical (open) half-plane $H_{u,1}(t_0) \subset \{(x,t) \in \mathbb{R}^3 : x_1 \leq 0\}$ such that for any $\delta > 0$, there is a $\rho > 0$ and a function*

$$w_{u,1} : H_{u,1}(t_0) \cap B_\rho((0,t_0)) \to \mathbb{R}$$

*a solution to the MSE over $H_{u,1}(t_0) \cap B_\rho((0,t_0))$ such that*

$$G_{u,1} \cap B_\rho((0,t_0)) = L_{u,1}(t_0)$$

*where*

$$L_{u,1}(t_0) = \left(\operatorname{graph}_{H_{u,1}(t_0)\cap B_\rho((0,t_0))} w_{u,1}\right) \cap B_\rho((0,t_0))$$

*with* $\operatorname{graph}_{H_{u,1}(t_0)\cap B_\rho((0,t_0))} w_{u,1}$

$$=\left\{\left(x + w_{u,1}((x,t))\nu_{H_{u,1}(t_0)}, t\right) : (x,t) \in H_{u,1}(t_0) \cap B_\rho((0,t_0))\right\}$$

*for $\nu_{H_{u,1}(t_0)}$ the unit normal of $H_{u,1}(t_0)$ oriented counterclockwise, $G_{u,1} = \operatorname{graph}_{\Omega_0} u_1$ for the component function $u_1(r,\theta) = u_0(r^{1/2}, \theta/2)$, and*

$\Omega_0 = \{re^{i\theta} : r \in (0,1), \theta \in (0, 2\pi)\}$. *Furthermore,* $w_{u,1}$ *has small scaled* $C^1$ *norm*

$$\frac{|w_{u,1}((x,t))|}{\rho} + |Dw_{u,1}((x,t))| < \delta$$

*for* $(x,t) \in H_{u,1}(t_0) \cap B_\rho((0,t_0))$.

We claim that in applying Lemma 10.2 to $u_0, U_0$ at $t_u, t_U$ respectively, we must have $H_{u,1}(t_u) = H_{U,1}(t_U)$. To show this, suppose for contradiction $H_{u,1}(t_u) \neq H_{U,1}(t_U)$, and write

$$H_{u,1}(t_u) = \{\theta = \theta_{u,1}(t_u)\}, \; H_{U,1}(t_U) = \{\theta = \theta_{U,1}(t_U)\}$$

where $\theta_{u,1}(t_u), \theta_{U,1}(t_U) \in [\pi/2, 3\pi/2]$ are not equal. We take without loss of generality the case $\theta_{U,1}(t_U) < \theta_{u,1}(t_u)$.

Since $\Phi(\varphi_0) = -1$, then $\theta_{u,1}(b) = \pi/2$ by Theorem 4.5, implying $t_u \neq b$. Moreover, the angles $\theta_{u,1}(t)$ decrease as $t \in [\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0]$ increases, so we can take $t_0 \in (t_u, b)$ such that

$$\theta_{U,1}(t_U) < \theta_{u,1}(t_0) < \theta_{u,1}(t_u).$$

Taking, for $\delta > 0$ small to be chosen later, $\rho > 0$ sufficiently small so that

$$G_{u,1} \cap B_\rho((0,t_0)) = L_{u,1}(t_0)$$

as in Lemma 10.2, then

$$G_{u,1} \cap B_\rho((0,t_0)) \cap \{(x,t_0) : x \in \mathcal{D}_\rho\} = \mathrm{graph}_{\gamma_{u,t_0}} u_1$$

where $\gamma_{u,t_0} \subset \Omega_0$ is a smooth curve asymptotic to the ray $\{re^{i\theta_{u,1}(t_0)} : r > 0\}$ at the origin. As a consequence

$$(\Omega_0 \cap \mathcal{D}_\rho) \setminus \gamma_{u,t_0} = O^{>t_0} \cup O^{<t_0},$$

where

$$O^{>t_0} = \{re^{i\theta} \in \Omega_0 \cap \mathcal{D}_\rho : u_1(r,\theta) > t_0\}, O^{<t_0} = \{re^{i\theta} \in \Omega_0 \cap \mathcal{D}_\rho : u_1(r,\theta) < t_0\}$$

are each connected open sets.

Since $\theta_{U,1}(t_U) < \theta_{u,1}(t_0)$, then $\Phi(\varphi_0) = -1$ implies that for $\delta > 0$ sufficiently small (depending on $\theta_{u,1}(t) - \theta_{U,1}(t_U)$)

$$\gamma_{U,t_U} \subset O^{>t_0}$$

where $\gamma_{U,t_U}$ is the smooth curve asymptotic to the ray $\{re^{i\theta_{U,1}(t_U)} : r > 0\}$ at the origin so that

$$G_{U,1} \cap B_\rho((0,t_U)) \cap \{(x,t_U) : x \in \mathcal{D}_\rho\} = \mathrm{graph}_{\gamma_{U,t_U}} U_1.$$

On the other hand, this now implies that for $re^{i\theta} \in \gamma_{U,t_U} \cap \Omega_0$,

$$U_1(r,\theta) - u_1(r,\theta) < t_U - t_0 < t_U - t_u = \varliminf_{r\to 0}(U_0 - u_0),$$

which gives a contradiction.

We hence have as claimed $H_{u,1}(t_u) = H_{U,1}(t_U)$ whenever $t_U - t_u = \varliminf_{r\to 0}(U_0 - u_0)$. We also claim we can assume $\varliminf_{r\to 0}(U_0 - u_0) < 0$ without loss of generality. For this, observe that $\varliminf_{r\to 0}(U_0 - u_0) = \varlimsup_{r\to 0}(U_0 - u_0) = 0$ implies $H_{u,1}(t) = H_{U,1}(t)$ for each $t \in (\varliminf_{r\to 0} u_0, \varlimsup_{r\to 0} u_0) = (\varliminf_{r\to 0} U_0, \varlimsup_{r\to 0} U_0)$. Uniqueness of the Cauchy problem, applied to $w_{u,1}, w_{U,1}$ at some $t \in (\varliminf_{r\to 0} u_0, \varlimsup_{r\to 0} u_0)$, would then give $u_0 = U_0$. This is a contradiction, since $\varphi_0 \neq P_0$, and so either $\varliminf_{r\to 0}(U_0 - u_0) \neq 0$ or $\varlimsup_{r\to 0}(U_0 - u_0) \neq 0$.

Assuming that $\varliminf_{r\to 0}(U_0 - u_0) < 0$, then take $t_u \in (\varliminf_{r\to 0} u_0, \varlimsup_{r\to 0} u_0]$ and $t_U \in (\varliminf_{r\to 0} U_0, \varlimsup_{r\to 0} U_0]$ (necessarily $t_U \neq A$) with $t_U - t_u = \varliminf_{r\to 0}(U_0 - u_0)$, and write $H = H_{u,1}(t_u) = H_{U,1}(t_U)$. Choose $t_0 \in (t_U, t_u) \cap (\varliminf_{r\to 0} u_0, \varlimsup_{r\to 0} u_0)$, and choose $\rho_0$ small so that

$$G_{u,1} \cap B_{\rho_0}((0,t_0)) = L_{u,1}(t_0)$$

as in Lemma 10.2. Considering again $G_{u,1} \cap B_{\rho_0}((0,t_0)) \cap \{(x,t_0) : x \in \mathcal{D}_{\rho_0}\}$, then

$$O^{>t_0} = \{re^{i\theta} \in \Omega_0 \cap \mathcal{D}_{\rho_0} : u_1(r,\theta) > t_0\}$$

is a connected open set with

$$(\partial O^{>t_0}) \cap \mathcal{D}_{\rho_0} = \{(x_1, 0) : x_1 \in [0,\rho_0)\} \cup \gamma_{u,t_0}$$

where $\gamma_{u_1,t_0}$ is the continuous curve asymptotic at the origin to the ray $\{re^{i\theta_0} : r > 0\}$ so that

$$G_{u,1} \cap B_{\rho_0}((0,t_0)) \cap \{(x,t_0) : x \in \mathcal{D}_{\rho_0}\} = \mathrm{graph}_{\gamma_{u,t_0}} u_1.$$

On the other hand, $\Phi(\varphi_0) = -1$ and $t_0 < t_u$ imply

$$\theta_{U,1}(t_U) = \theta_{u,1}(t_u) < \theta_{u,1}(t_0).$$

We can thus choose $\delta > 0$ sufficiently small, depending on $\theta_{u,1}(t_0) - \theta_{u,1}(t_u)$, and $\rho > 0$ corresponding so that

$$G_{U,1} \cap B_\rho((0, t_U)) = L_{U,1}(t_U) \subseteq O^{>t_0}$$

by Lemma 10.2. We therefore have for $re^{i\theta} \in \operatorname{proj}_{\Omega_0} L_{U,1}(t_U)$

$$U_1(r, \theta) - u_1(r, \theta) < t_U + \rho - t_0 < 0,$$

if we suppose $\rho < t_0 - t_U$ as well. We have therefore shown that $\operatorname{proj}_{\Omega_0} L_{U,1}(t_U) \subseteq \{re^{i\theta} \in \Omega_0 : U_1(r, \theta) - u_1(r, \theta) < 0\}$.

We claim that there are points $re^{i\theta} \in \operatorname{proj}_{\Omega_0} L_{U,1}(t_U)$ so that

$$U_1(r, \theta) - u_1(r, \theta) < t_U - t_u.$$

To show this, first consider the function $\tilde{w}_{U,1} : H \cap B_\rho((0, t_u)) \to \mathbb{R}$ given by

$$\tilde{w}_{U,1}((x, t)) = w_{U,1}((x, t + t_U - t_u)).$$

Then $H = H_{u,1}(t_u) = H_{U,1}(t_U)$ and the Hopf Boundary Point Lemma, applied to $w_{u,1}, \tilde{w}_{U,1}$ at the point $(0, t_u)$, imply that we cannot have $w_{u,1} \leq \tilde{w}_{U,1}$ in $H \cap B_{\rho/4}((0, t_u))$.

Take thus a point $(\tilde{x}, \tilde{t}) \in H \cap B_{\rho/4}((0, t_u))$ such that

$$w_{U,1}((\tilde{x}, \tilde{t} + t_U - t_u)) = \tilde{w}_{U,1}((\tilde{x}, \tilde{t})) < w_{u,1}((\tilde{x}, \tilde{t})).$$

Then by definition of $w_{u,1}, w_{U,1}$

$$u_1\Big(\tilde{x} + w_{u,1}((\tilde{x}, \tilde{t}))\nu_H\Big) = \tilde{t}$$

and

$$U_1\Big(\tilde{x} + w_{U,1}((\tilde{x}, \tilde{t} + t_U - t_u))\nu_H\Big) = \tilde{t} + t_U - t_u,$$

where $\nu_H$ is the unit normal of $H = H_{u,1}(t_u) = H_{U,1}(t_U)$ oriented counterclockwise (observe that since $\theta_{u,1}(t_u) = \theta_{U,1}(t_U) \in (\pi/2, 3\pi/2]$ and $w_{u,1}, w_{U,1}$ both have scaled $C^1$ norm less than $\delta$, then $\tilde{x} + w_{u,1}((\tilde{x}, \tilde{t})\nu_H$ and $\tilde{x} + w_{U,1}((\tilde{x}, \tilde{t} + t_U - t_u))\nu_H$ are both in $\Omega_0$).

Now, $\Phi(\varphi_0) = \Phi(P_0) = -1$ implies $Du_1(r, \theta) \cdot \nu_H < 0$ for each $re^{i\theta} \in \operatorname{proj}_{\Omega_0} L_{u,1}(t_u)$ and $DU_1(r, \theta) \cdot \nu_H < 0$ for each $re^{i\theta} \in \operatorname{proj}_{\Omega_0} L_{U,1}(t_U)$. Since $w_{U,1}((\tilde{x}, \tilde{t} + t_U - t_u)) < w_{u,1}((\tilde{x}, \tilde{t}))$ and

$$u_1\Big(\tilde{x} + w_{u,1}((\tilde{x}, \tilde{t}))\nu_H\Big) = \tilde{t}$$

and

$$U_1\Big(\tilde{x} + w_{U,1}((\tilde{x}, \tilde{t} + t_U - t_u))\nu_H\Big) = \tilde{t} + t_U - t_u,$$

then we can find a $w \in \Big(w_{U,1}((\tilde{x}, \tilde{t} + t_U - t_u)), w_{u,1}((\tilde{x}, \tilde{t}))\Big]$ so that

$$u_1(\tilde{x} + w\nu_H) > \tilde{t}$$

while

$$U_1(\tilde{x} + w\nu_H) \in (\tilde{t} + t_U - t_u - \rho/4, \tilde{t} + t_U - t_u]$$

(note that $\tilde{x} + w\nu_H \in \Omega_0$ for each $w \in (w_{U,1}((\tilde{x}, \tilde{t} + t_U - t_u)), w_{u,1}((\tilde{x}, \tilde{t}))])$. Observe that

$$U_1(\tilde{x} + w\nu_H) - u_1(\tilde{x} + w\nu_H) < t_U - t_u.$$

Furthermore,

$$\Big|\Big(\tilde{x} + w\nu_H, U_1(\tilde{x} + w\nu_H)\Big) - (0, t_U)\Big| \leq |\tilde{x}| + |w| + |\tilde{t} - t_u| + \rho/4,$$

which since $(\tilde{x}, \tilde{t}) \in B_{\rho/4}((0, t_u))$ and $w_{u,1}, w_{U,1}$ have scaled $C^1$ norm less than $\delta$ implies

$$\Big|\Big(\tilde{x} + w\nu_H, U_1(\tilde{x} + w\nu_H)\Big) - (0, t_U)\Big| < \rho/4 + \delta\rho + \rho/4 + \rho/4 < \rho$$

for $\delta < 1/4$. We therefore have

$$(\tilde{x} + w\nu_H, U_1(\tilde{x} + w\nu_H)) \in G_{U,1} \cap B_\rho((0, t_U)),$$

so that $\tilde{x} + w\nu_H \in \operatorname{proj}_{\Omega_0} L_{U,1}(t_U)$.

We have hence shown $\operatorname{proj}_{\Omega_0} L_{U,1}(t_U) \subseteq \{re^{i\theta} \in \Omega_0 : U_1(r, \theta) - u_1(r, \theta) < 0\}$ and that there are points $re^{i\theta} \in \operatorname{proj}_{\Omega_0} L_{U,1}(t_U)$ so that

$$U_1(r, \theta) - u_1(r, \theta) < t_U - t_u = \varliminf_{r \to 0}(U_0 - u_0).$$

Consider the set

$$\sqrt{\operatorname{proj}_{\Omega_0} L_{U,1}(t_U)} = \{re^{i\theta} \in \mathcal{D} : \theta \in (0, \pi), r^2 e^{i2\theta} \in \operatorname{proj}_{\Omega_0} L_{U,1}(t_U)\}.$$

Since $U_0(r, \theta) = U_1((r^2)^{1/2}, (2\theta)/2)$ for $\theta \in (0, \pi)$, then

$$\sqrt{\operatorname{proj}_{\Omega_0} L_{U,1}(t_U)} \subseteq \{x \in \mathcal{D} : U_0(x) - u_0(x) < 0\}$$

and there are points $\tilde{x} \in \sqrt{\operatorname{proj}_{\Omega_0} L_{U,1}(t_U)}$ such that $U_0(\tilde{x}) - u_0(\tilde{x}) < t_U - t_u$. Furthermore, (for $\rho > 0$ sufficiently small)

$\sqrt{\mathrm{proj}_{\Omega_0} L_{U,1}(t_U)}$ is connected, and so we let $O^-$ be the connected component of $\{x \in \mathcal{D} : U_0(x) - u_0(x) < 0\}$ containing $\sqrt{\mathrm{proj}_{\Omega_0} L_{U,1}(t_U)}$.

Observe that since $t_U - t_u = \underline{\lim}_{r\to 0}(U_0 - u_0)$, then $U_0 - u_0$ cannot achieve its minimum in $O^-$ at the origin, and so we must also have

$$\partial O^- \cap \{e^{i\theta} : P_0(\theta) - \varphi_0(\theta) < 0\} \neq \emptyset.$$

Let

$$\sqrt{\gamma_{U,t_U}} = \{re^{i\theta} \in \mathcal{D} : \theta \in (0,\pi), r^2 e^{i2\theta} \in \gamma_{U,t_U}\}$$

where $\gamma_{U,t_U} \subset \Omega_0$ is the continuous curve so that

$$G_{U,1} \cap B_\rho((0,t_U)) \cap \{(x,t_U) : x \in \mathcal{D}_\rho\} = \mathrm{graph}_{\gamma_{U,t_U}} U_1.$$

Observe that $\sqrt{\gamma_{U,t_U}}$ is also a continuous curve, which is asymptotic at the origin to the ray $\{re^{i\theta_{U,1}(t_U)/2} : r > 0\}$ where $\theta_{U,1}(t_U)/2 \in (\pi/4, \frac{3\pi}{4}]$.

Since $O^-$ is connected, we can then find a continuous curve $\gamma^- \subset O^-$ with $\gamma^-(0) = 0$, $\gamma^-(1) \in \{e^{i\theta} : P_0(\theta) - \varphi_0(\theta) < 0\}$, and so that $\gamma^-(s) = \sqrt{\gamma_{U,t_U}}(s)$ for all $s > 0$ sufficiently small. Given that $u_0, U_0$ are both even in $\theta$, we can also choose the curve $\gamma^-$ so that $\gamma^-((0,1)) \subset \{x \in \mathcal{D} : x_2 > 0\}$.

Letting $R(x) = (x_1, -x_2)$ be the reflection across the $x_1$-axis, we have

$$\mathcal{D} \setminus \Big(\gamma^-([0,1)) \cup R(\gamma^-([0,1)))\Big) = W^- \cup W^+$$

where $W^-, W^+$ are connected open sets containing the negative and positive $x_1$-axis respectively. Since $P_0$ is steeper than $\varphi_0$, let $\Theta_0 \in (0,\pi)$ be given so that $P_0(\theta) - \varphi_0(\theta) \geq 0$ for $\theta \in (-\Theta_0, \Theta_0)$ while $P_0(\theta) - \varphi_0(\theta) \leq 0$ for $\theta \in (\Theta_0, 2\pi - \Theta_0)$. Recalling that $\gamma^-(1) \in \{e^{i\theta} : P_0(\theta) - \varphi_0(\theta) < 0\}$, then

$$\partial W^- \cap \partial\mathcal{D} \subset \{e^{i\theta} : \theta \in [\Theta_0, 2\pi - \Theta_0]\},$$

and hence $(\partial W^-) \setminus \{0\} \subset \{x \in \overline{\mathcal{D}} : U_0(x) - u_0(x) \leq 0\}$.

Recalling our assumption $[\underline{\lim}_{r\to 0} U_0, \overline{\lim}_{r\to 0} U_0] \subseteq [\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0]$, then $A - a \geq 0$ for $A = \underline{\lim}_{r\to 0} U_0, a = \underline{\lim}_{r\to 0} u_0$. Since $\Phi(\varphi_0) = \Phi(P_0) = -1$, then by Lemma 6.1

$$\underline{\lim}_{r\to 0}(U_0(r,\pi) - u_0(r,\pi)) = A - a,$$

so that $\overline{\lim}_{x\to 0, x\in W^-}(U_0 - u_0) \geq 0$. We can then choose $t^u \in [\underline{\lim}_{r\to 0} u_0, t_u), t^U \in [\underline{\lim}_{r\to 0} U_0, t_U)$ so that

$$t^U - t^u = \overline{\lim_{x\to 0, x\in W^-}}(U_0 - u_0),$$

where again we must have $H_{u,1}(t^u) = H_{U,1}(t^U)$ where $H_{u,1}(t^u), H_{U,1}(t^U)$ are the half-planes in the statement of Lemma 10.2 applied to $u_1, U_1$ at the heights $t^u, t^U$ respectively.

Since $t^U < t_U, t^u < t_u$ implies $\theta_{u,1}(t^u) = \theta_{U,1}(t^U) > \theta_{u,1}(t_u) = \theta_{U,1}(t_U)$, we can use the Hopf Boundary Point Lemma to show there are points $\tilde{x} \in W^-$ such that

$$U_0(\tilde{x}) - u_0(\tilde{x}) > t^U - t^u \geq 0.$$

Let $O^+$ be a connected component of $\{x \in \mathcal{D} : U_0(x) - u_0(x) > 0\}$ containing such an $\tilde{x}$. Since $(\partial W^- \setminus \{0\}) \subseteq \{x \in \mathcal{D} : U_0(x) - u_0(x) \leq 0\}$, then $O^+ \subset W^-$ and $(\partial O^+ \setminus \{0\}) \subseteq \{x \in \mathcal{D} : U_0(x) - u_0(x) = 0\}$. This means that $U_0 - u_0$ must achieve its maximum in $O^+$ at the origin, which contradicts $\tilde{x} \in O^-$ and $t^U - t^u = \overline{\lim}_{x\to 0, x\in W^-}(U_0 - u_0)$. $\square$

# 11 q-Valued Minimal Surface Equation

The results of the previous sections can be extended to functions of more than two values. For $q \geq 2$, in [SW07] it is noted that given $\varphi_0 \in C(S_1)$ we can solve the equation

$$\mathcal{M}_{q,0}(v) = \sum_{i=1}^{2} D_{x_i}\left(\frac{D_{x_i}v}{\sqrt{1 + (qr^{q-1})^{-2}|D_x v|^2}}\right)$$

in $\mathcal{D}$ with boundary data $\varphi_0$. This gives us a unique solution $u_0$ so that the $q$-valued graph

$$G = \{(re^{i\theta}, u_0(r^{1/q}, \theta/q)) : r \in (0,1), \theta \in [0, 2q\pi)\}$$

is an immersed minimal surface with prescribed boundary data $\varphi_0(\theta/q)$. It thus makes sense to call $\mathcal{M}_{q,0}$ the $q$-valued Minimal Surface Equation, or $q$MSE. As before, we write $u(r,\theta) = u_0(r^{1/q}, \theta/q)$.

In this case, the analogue of Theorem 4.4 we shall need is the following:

**Theorem 11.1** *Suppose that $\varphi_0$ is even in $\theta$, so that $\varphi_0(-\theta) = \varphi_0(\theta)$. Suppose also $\frac{q-1}{2} < k < q$ with $(k,q) = 1$, and $\varphi_0$ is $\frac{2\pi}{k}$ rotationally* *symmetric, so that $\varphi_0(\theta + \frac{2\pi}{k}) = \varphi_0(\theta)$. If the solution $u_0$ to the qMSE with boundary data $\varphi_0$ cannot be extended continuously across the origin, then for every $t \in (\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0)$ there is a $\rho > 0$ such that*

$$G \cap B_\rho((0,t)) = \bigcup_{j=1}^{2k} L_j(t)$$

*where each $L_j(t)$ is an embedded minimal surface-with-boundary $\{0\} \times (t-\rho, t+\rho)$. We can further label the surfaces so that*

$$L_j(t) \subset \left\{ (re^{i\theta}, u(r,\theta)) : r \in (0,\rho), \theta \in \left( \frac{(j-1)q\pi}{k} + \frac{\pi}{2}, \frac{jq\pi}{k} - \frac{\pi}{2} \right) \right\}$$

*for $j = 1, \dots, 2k$. Furthermore, $G$ has unique tangent cones at $(0, \underline{\lim}_{r\to 0} u_0), (0, \overline{\lim}_{r\to 0} u_0)$, each consisting of $k$ vertical planes (counted with multiplicity).*

**Proof:** Theorem 12.1 of [R07] gives a general description of the $q$-valued graph $G$ near $\{0\} \times [\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0]$ of a solution $u_0$ to the $q$MSE which cannot be extended continuously across the origin (with no symmetry assumptions on the boundary data). That $G \cap B_\rho((0,t)) = \bigcup_{j=1}^{2k} L_j(t)$ for each $t \in (\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0)$ when the boundary data is even and $\frac{2\pi}{k}$ rotationally symmetric as above, is shown in the proof of the second part of Theorem 12.2 of [R07]. Theorem 12.1 of [R07] also gives a general description of the tangent cones of $G$ at $(0, \underline{\lim}_{r\to 0} u_0), (0, \overline{\lim}_{r\to 0} u_0)$, stating that they respectively consist of at most $q-1$ vertical planes.

As we will need a more specific description of the tangent cones of $G$ at $(0, \underline{\lim}_{r\to 0} u_0), (0, \overline{\lim}_{r\to 0} u_0)$, let $t \in (\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0)$ and consider the surface $L_j(t)$ as above. Then we can continuously choose $\theta_j(t) \in \left( \frac{(j-1)q\pi}{k} + \frac{\pi}{2}, \frac{jq\pi}{k} - \frac{\pi}{2} \right)$ such that $H_j(t) = \{\theta = \theta_j(t)\}$, where $H_j(t)$ is the tangent vertical half-plane of $L_j(t)$ at $(0,t)$. We then define $\Phi(\varphi_0) = -1$ if $\theta_1(t)$ is decreasing as $t \in (\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0)$ increases, and $\Phi(\varphi_0) = 1$ if instead $\theta_1(t)$ is increasing (we also define $\Phi(\varphi_0) = 0$ in case $u_0$ can be extended continuously across the origin).

Suppose that $\Phi(\varphi_0) = -1$, and let $\theta_j(b) \in \left[ \frac{(j-1)q\pi}{k} + \frac{\pi}{2}, \frac{jq\pi}{k} - \frac{\pi}{2} \right]$ be the limit of the angles $\theta_j(t)$ as $t$ increases to $b = \overline{\lim}_{r\to 0} u_0$. The proof of Theorem 12.1 of [R07] shows that the unique tangent cone $\mathbb{C}$ of $G$ at $(0,b)$ can be written as $\mathbb{C} = \sum_{\ell=0}^{k-1} \mathbb{C}_\ell$ where

$$\mathbb{C}_0 = |\{\theta = \theta_{2k}(b)\}| + |\{\theta = \theta_{2k}(b) + \pi\}| + \dots$$

$$\ldots + |\{\theta = \theta_{2k}(b) + \pi n_0 = \theta_1(b) + 2q\pi\}|,$$

and for $1 \leq \ell \leq k-1$

$$\mathbb{C}_\ell = |\{\theta = \theta_{2\ell}(b)\}| + |\{\theta = \theta_{2\ell}(b) + \pi\}| + \ldots$$

$$\ldots + |\{\theta = \theta_{2\ell}(b) + \pi n_\ell = \theta_{2\ell+1}(b)\}|$$

where each $n_\ell \in \mathbb{N}$ is odd, and $|\{\theta = \theta_0\}|$ is the multiplicity one two-dimensional varifold associated to the half-plane $\{\theta = \theta_0\}$. Furthermore, for each $0 \leq \ell \leq k-1$, $\mathbb{C}_\ell$ is the unique tangent cone of

$$\left\{(re^{i\theta}, u(r,\theta)) : r \in (0,1), \theta \in \left(\frac{(2\ell-1)q\pi}{k} + \frac{\pi}{2}, \frac{(2\ell+1)q\pi}{k} - \frac{\pi}{2}\right)\right\}$$

at $(0,b)$.

This shows $\mathbb{C}$ consists of at least $k$ many vertical planes. On the other hand, $u(r, \theta + \frac{2q\pi}{k}) = u(r,\theta)$, which implies that for each $1 \leq \ell \leq k-1$

$$\mathbb{C}_\ell = S_{\frac{2\ell q\pi}{k}}(\mathbb{C}_0)$$

where $S_{\frac{2\ell q\pi}{k}}$ is the rotation $S_{\frac{2\ell q\pi}{k}}((re^{i\theta}, t)) = (re^{i(\theta + \frac{2\ell q\pi}{k})}, t)$. Hence, if $\mathbb{C}_0$ does not consist of a single vertical plane, then $\mathbb{C}$ must consist of at least $2k$ many vertical planes. This is a contradiction, since $\frac{q-1}{2} < k$ and $\mathbb{C}$ cannot consist of more than $q-1$ many vertical planes.

Hence $\mathbb{C} = \sum_{j=1}^{2k} |\{\theta = \theta_j(b)\}|$, and similarly, $G$ has unique tangent cone $\sum_{j=1}^{2k} |\{\theta = \theta_j(a)\}|$ at $(0,a)$ for $a = \underline{\lim}_{r\to 0} u_0$. We can now state the $q$-valued versions of our main results Theorems 5.2, 5.3, 5.4. First, we make an appropriate definition:

**Definition 11.2** *Suppose $\varphi_0, P_0 \in C(\partial\mathcal{D})$ are both even boundary data which are $\frac{2\pi}{k}$ rotationally symmetric for $k \in \mathbb{N}$, then we say that $P_0$ is $k$-steeper than $\varphi_0$ if there exists a $\Theta_0 \in (0, \frac{\pi}{k})$ such that*

$$P_0(\theta) \geq \varphi_0(\theta) \textit{ for } \theta \in [0, \Theta_0)$$
$$P_0(\theta) \leq \varphi_0(\theta) \textit{ for } \theta \in (\Theta_0, \frac{\pi}{k}].$$

The following extensions then hold:

**Theorem 11.3** *Suppose $\frac{q-1}{2} < k < q, (q,k) = 1$ and $\varphi_0, P_0 \in C(\partial\mathcal{D})$ are both even $\frac{2\pi}{k}$ rotationally symmetric boundary data with $P_0$ $k$-steeper than $\varphi_0$. Also suppose $\Phi(\varphi_0) = -1$ and that, for $u_0, U_0$ the solutions to the 2MSE with boundary data $\varphi_0, P_0$ respectively,*

$$[\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0] \cap [\underline{\lim}_{r\to 0} U_0, \overline{\lim}_{r\to 0} U_0] \neq \emptyset.$$

*Then $\Phi(P_0) = -1$ as well.*

As in the two-valued case, this is used to prove the following converse to the second part of Theorem 12.2 of [R07]:

**Theorem 11.4** *Suppose $\frac{q-1}{2} < k < q, (q,k) = 1$ and $\varphi_0, \psi_0 \in C(\partial\mathcal{D})$ are both even $\frac{2\pi}{k}$ rotationally symmetric boundary data with $\Phi(\varphi_0) = \Phi(\psi_0) \neq 0$. Then there is a continuous deformation $\varphi_{s,0}$ of even $\frac{2\pi}{k}$ rotationally symmetric continuous functions with $\varphi_{0,0} = \varphi_0$ and $\varphi_{1,0} = \psi_0$ such that for each $s \in [0,1]$ we have $\Phi(\varphi_{s,0}) = \Phi(\varphi_0) = \Phi(\psi_0)$.*

We also have a version of Theorem 5.4:

**Theorem 11.5** *Suppose $\frac{q-1}{2} < k < q, (q,k) = 1$, then for all $\vartheta_0 \in (0, \frac{\pi}{k})$ and $M > 0$, there exists an $\epsilon > 0$ so that $\Phi(\varphi_0) = -1$ for all even $\frac{2\pi}{k}$ rotationally symmetric boundary data $\varphi_0 \in C(\partial\mathcal{D})$ with $\|\varphi_0\|_{C(\partial\mathcal{D})} < M + \epsilon$, $\varphi_0(\theta) \geq M$ for $\theta \in [0, \vartheta_0 - \epsilon]$, and $\varphi_0(\theta) \leq -M$ for $\theta \in [\vartheta_0 + \epsilon, \frac{\pi}{k}]$.*

**Proofs:** Theorem 11.5 follows from Theorem 11.3 as in the two-valued case, using the comparison boundary data $\varphi_{\alpha,0} = \alpha\cos(k\theta)$ for $\alpha > 0$ which yield discontinuous solutions to the $q$MSE by Theorem 13.1 of [R07] (the proof that $\Phi(\varphi_{\alpha,0}) = -1$ follows similarly as the proof that $\Phi(\alpha\cos(\theta)) = -1$ in the two-valued case). Theorem 11.4 also follows from Theorem 11.3 through a modification of the proof of Theorem 5.3.

To more clearly see how to modify the arguments of Theorems 5.3,5.4 to the present $q$-valued case, we describe the proof of Theorem 11.3. Assuming that $\Phi(\varphi_0) = -1, \Phi(P_0) \neq -1$ with

$$[\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0] \cap [\underline{\lim}_{r\to 0} U_0, \overline{\lim}_{r\to 0} U_0] \neq \emptyset,$$

we can show as in the two-valued case that there is a $\sigma > 0$ sufficiently small so that

$$U_0(r,0) - u_0(r,0) < \underline{\lim}_{r\to 0} U_0 - \overline{\lim}_{r\to 0} u_0 \leq 0$$

using the fact that $\theta_{u_0,1}(b) = \frac{\pi}{2}$ where $b = \overline{\lim}_{r\to 0} u_0$. If we consider $O^-$ to be the connected component of $\{x \in \mathcal{D} : U_0(x) - u_0(x) < 0\}$ containing $(0,\sigma) \times \{0\}$, then the Maximum Principle implies that $\partial O^- \cap \{e^{i\theta} : P_0(\theta) - \varphi_0(\theta) < 0\} \neq \emptyset$.

Since $u_0$ is even and $\frac{2\pi}{k}$ rotationally symmetric in $\theta$, then $u_0$ is symmetric about $\theta = \frac{\pi\ell}{k}$ for each $\ell = 0, \ldots, 2k-1$. We can thus find an $\ell^- \in \{0, \ldots, \lfloor\frac{k-1}{2}\rfloor\}$ and a $\theta^- \in (\Theta_0 + \frac{2\pi\ell^-}{k}, \frac{\pi}{k} + \frac{2\pi\ell^-}{k})$ (note $\theta^- \in (0,\pi)$) along with a continuous curve $\gamma^- \subset \{x \in \mathcal{D} : U_0(x) - u_0(x) < 0\}$ with $\gamma^-((0,1)) \subset \{x \in \mathcal{D} : x_2 \geq 0\}$ such that $\gamma^-(s) = (s,0)$ for $s \in [0,\sigma)$, and $\gamma^-(1) = e^{i\theta^-}$.

We can similarly find an $\ell^+ \in \{0, \ldots, \lfloor\frac{k-1}{2}\rfloor\}$ and a $\theta^+ \in (\frac{2\pi\ell^+}{k}, \Theta_0 + \frac{2\pi\ell^+}{k})$ along with a continuous curve $\gamma^+ \subset \{x \in \mathcal{D} : U_0(x) - u_0(x) > 0\}$ with $\gamma^+((0,1)) \subset \{x \in \mathcal{D} : x_2 \geq 0\}$ such that $\gamma^+(s) = (-s,0)$ for $s \in [0,\sigma)$, and $\gamma^+(1) = e^{i\theta^+}$.

If $\ell^+ \leq \ell^-$, then the curves $\gamma^-$ and $\gamma^+$ must intersect, giving a contradiction. Assuming $\ell^+ > \ell^-$, then consider

$$\mathrm{Rot}(re^{i\theta}) = re^{i\left(\theta - \frac{2\pi(\ell^+ - \ell^-)}{k}\right)}.$$

However $u_0, U_0$ are $\frac{2\pi}{k}$ rotationally symmetric, which implies that $\mathrm{Rot}(\gamma^+) \subset \{x \in \mathcal{D} : U_0(x) - u_0(x) > 0\}$ as well as

$$\mathrm{Rot}(\gamma^+) \subset \left\{re^{i\theta} : r \in (0,1), \theta \in \left[-\frac{2\pi(\ell^+ - \ell^-)}{k}, \pi - \frac{2\pi(\ell^+ - \ell^-)}{k}\right]\right\}.$$

On the other hand, $\mathrm{Rot}(\gamma^+)(1) \in \{e^{i\theta} : \theta \in (\frac{2\pi\ell^-}{k}, \Theta_0 + \frac{2\pi\ell^-}{k})\}$, and

$$\mathrm{Rot}(\gamma^+)(s) = se^{i\left(\pi - \frac{2\pi(\ell^+ - \ell^-)}{k}\right)}$$

for $s \in (0,\sigma)$ where

$$0 < \pi - \frac{2\pi(\ell^+ - \ell^-)}{k} < \pi.$$

Since $\gamma^-(s) = (s,0)$ for $s \in [0,\sigma)$ and $\gamma^-(1) = e^{i\theta^-}$ with $\theta^- \in (\Theta_0 + \frac{2\pi\ell^-}{k}, \frac{\pi}{k} + \frac{2\pi\ell^-}{k})$, then $\mathrm{Rot}(\gamma^+)$ and $\gamma-$ must intersect, which is a contradiction. $\square$

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