

Discontinuous Solutions to the
Two-Valued Minimal Surface Equation

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0. Introduction

$M^n \subset \mathbb{R}^{n+1}$ immersed is stable if for X, ζ with compact support

$$\int_M \operatorname{div}_M X \, d\mathcal{H}^n = 0 \quad (\text{Minimal})$$

$$\int_M \zeta^2 |A_M|^2 \, d\mathcal{H}^n \leq \int_M |\nabla^M \zeta|^2 \, d\mathcal{H}^n \quad (\text{Stability Ineq}).$$

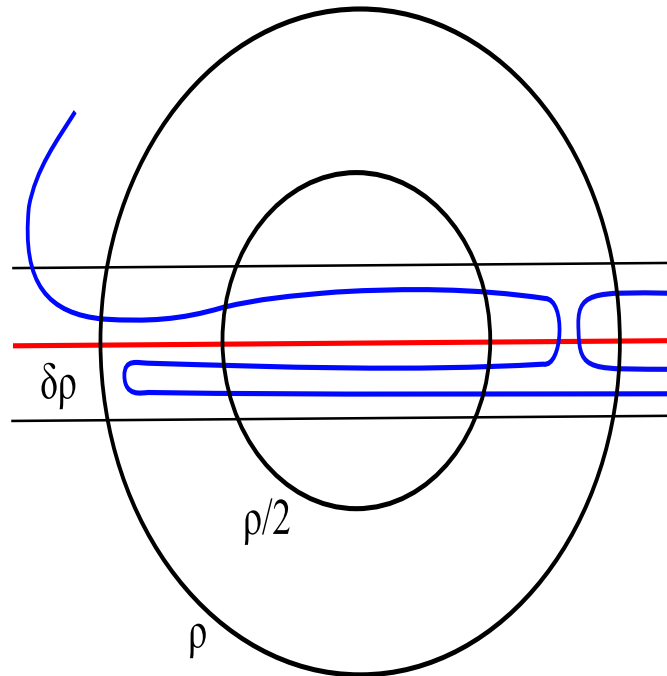
- Area minimizing M , and graphs of solutions to the MSE:

$$\sum_{j=1}^n D_{x_j} \left(\frac{Dx_j v}{\sqrt{1 + |Dv|^2}} \right) = 0$$

are stable.

- $\operatorname{sing} M =$ points near which \overline{M} is not smooth, embedded.

[Schoen-Simon81]: If $M^n \subset \mathbb{R}^{n+1}$ is an embedded, stable, with $\mathcal{H}^{n-2}(\text{sing}M) < \infty$, then



[Wickramasekera]: M^n immersed, then near $\Theta_M < 3$

$$\text{sing}M = \text{branch}M_{\leq n-2} \cup \mathbf{sing}M_{\leq n-7}.$$

[SW07]: PDE method for producing stable immersions $G^n \subset \mathbb{R}^{n+1}$ with branch set $(n - 2)$ -dim $C^{1,\alpha}$ submanifold, $\mathbb{Z}_{\frac{2\pi}{k}}$ symmetric.

[Rosales07]: For $n = 2$:

- More complete description of the PDE method.
- Give “even” examples of branched $G \subset \mathbb{R}^3$.

Thm: For $n = 2$, we further investigate the case when the PDE method does not produce branched immersions.

I. PDE Method

$$\mathcal{F}_0(v) = \int_{\mathcal{D}} 4r^2 \sqrt{1 + \frac{|Dv|^2}{4r^2}} dx$$

$$x = (x_1, x_2) = re^{i\theta} \text{ and } (x, t) \in \mathbb{R}^3$$

$$\mathcal{D} = \{|x| < 1\}, \mathcal{D}_\rho = \{|x| < \rho\}$$

$$\mathcal{M}_0(v) = \sum_{j=1}^2 D_{x_j} \left(\frac{D_{x_j} v}{\sqrt{1 + \frac{|Dv|^2}{4r^2}}} \right)$$

If

$$\mathcal{M}_0(u_0) = \sum_{j=1}^2 D_{x_j} \left(\frac{D_{x_j} u_0}{\sqrt{1 + \frac{|Du_0|^2}{4r^2}}} \right) = 0$$

Then $u(r, \theta) = u_0(r^{\frac{1}{2}} e^{i\theta/2})$ for $r \in (0, 1), \theta \in \mathbb{R}$

- locally solution to the MSE
- period 4π

$\mathcal{M}_0 = \text{Two-Valued MSE} = 2\text{MSE}$.

[SW07]: For all $\varphi_0 \in C(S^1)$, there is a unique $u_0 \in C^\infty(\mathcal{D} \setminus \{0\})$

$$\mathcal{M}_0(u_0) = \sum_{j=1}^2 D_{x_j} \left(\frac{D_{x_j} u_0}{\sqrt{1 + \frac{|Du_0|^2}{4r^2}}} \right) = 0,$$

- $u_0|_{S^1} = \varphi_0$ continuously,
- $\sup_{\mathcal{D}_\rho \setminus \mathcal{D}_\sigma} |Du_0| < C(\sigma, \rho, \sup_{S^1} |\varphi_0|)$,
- $\rho = 1$ if φ_0 is C^2 .

Proof:

$$\text{Let } r_\delta = \begin{cases} r & \text{if } r > \delta \\ \geq \delta/2 & \text{if } r < \delta \end{cases}$$

$$\mathcal{M}_\delta(v) = \sum_{j=1}^2 D_{x_j} \left(\frac{D_{x_j} v}{\sqrt{1 + \frac{|Dv|^2}{4r_\delta^2}}} \right) = 0.$$

- Interior and boundary gradient estimates for Non-Uniformly Elliptic PDE's by [S76] implies $\exists u_\delta|_{S^1} = \varphi_0$.
- A subsequence $u_\delta \rightarrow u_0$ uniformly on compact subsets of $\mathcal{D} \setminus \{0\}$.

Note that:

- $u_0 \in C(\mathcal{D} \setminus \{0\})$.
- Uniqueness follows as in [Finn53].
- Solutions and differences of solutions to the 2MSE satisfy the Maximum Principle.

We use \mathcal{M}_0 to get $C^{1,\alpha}$ branched stable surfaces.

- For $\mathcal{M}_0(u_0) = 0$, we define the two-valued graph

$$G := \{(re^{i\theta}, u(r, \theta)) : r \in (0, 1), \theta \in [0, 4\pi)\}$$

with $u(r, \theta) = u_0(r^{\frac{1}{2}}e^{i\theta/2})$.

- G is minimal in $(\mathcal{D} \times \mathbb{R}) \setminus (\{0\} \times \mathbb{R})$.

Stability depends on whether $u_0 \in C(\mathcal{D})$.

[SW07]: If $u_0 \in C(\mathcal{D})$, then G is $C^{1,\alpha}$ stable branched.

Proof:

- For $X = X_0(r^{\frac{1}{2}}e^{i\theta/2})$

$$\int_G \operatorname{div}_G X \, d\mathcal{H}^2 = 0$$

- Uniform Mass Bound: (for any G)

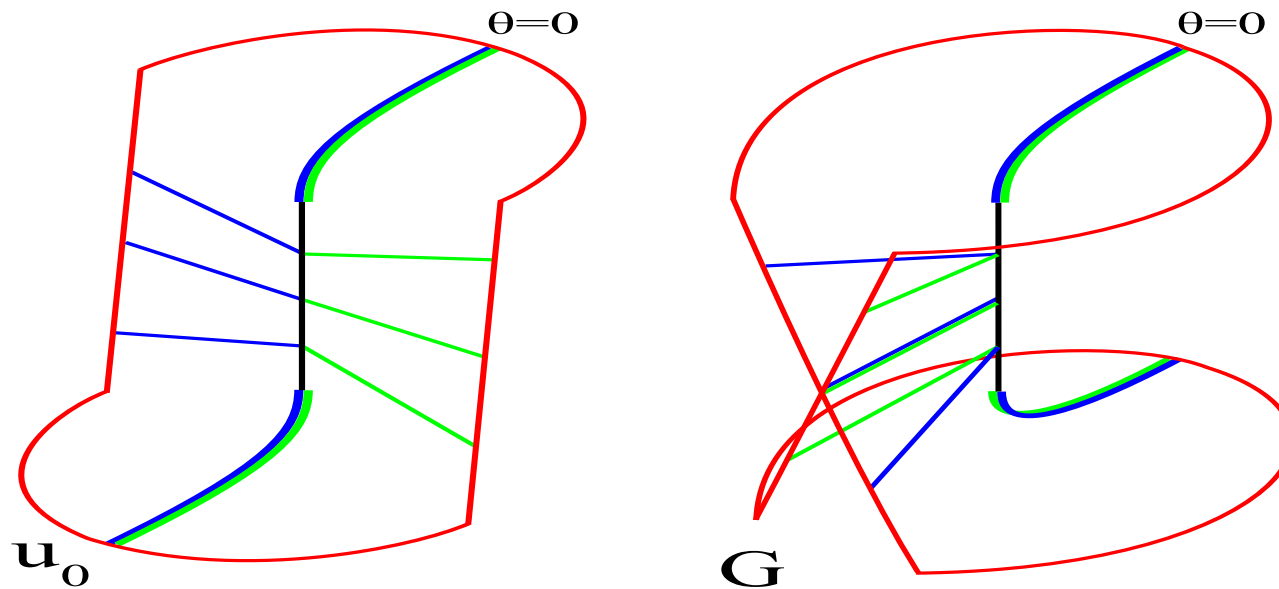
$$\mathcal{H}^2(G \cap B_\rho((x, t))) \leq c\rho^2$$

if $B_{2\rho}((x, t)) \subset \mathcal{D} \times \mathbb{R}$.

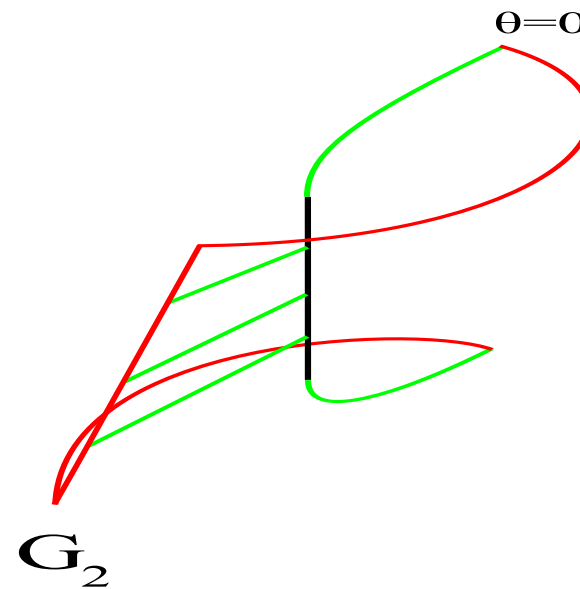
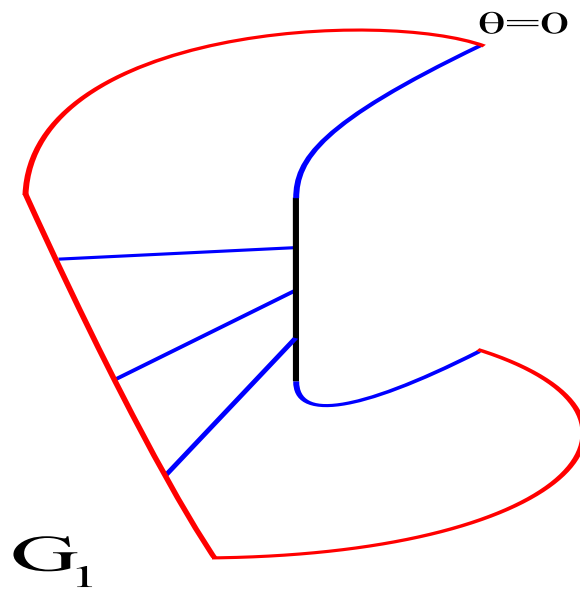
[SW07]: If $\varphi_0(\theta + \frac{2\pi}{k}) = \varphi_0(\theta)$ with $k > 1$ odd, then $u_0 \in C(\mathcal{D})$.

Take $u_0 \notin C(\mathcal{D})$, $G = \text{graph of } u(r, \theta) = u_0(r^{1/2} e^{i\theta/2})$.

- $\overline{G} \cap (\{0\} \times \mathbb{R}) = \{0\} \times [\underline{\lim}_{r \rightarrow 0} u_0, \overline{\lim}_{r \rightarrow 0} u_0]$



- interval of discontinuity $:= (\underline{\lim}_{r \rightarrow 0} u_0, \overline{\lim}_{r \rightarrow 0} u_0)$



- For $\theta_0 \in [0, 2\pi)$ then

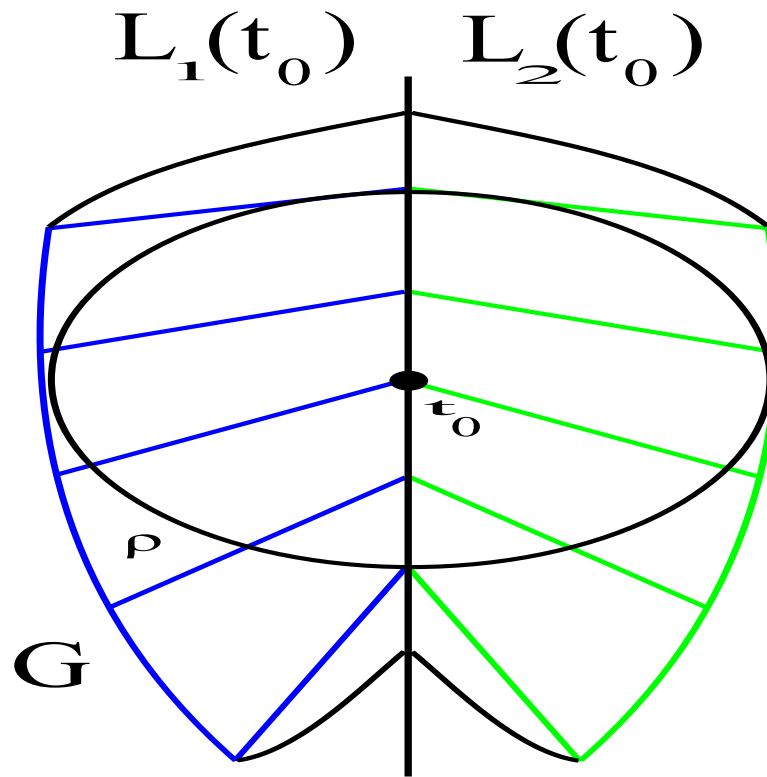
$$\begin{cases} u_1(r, \theta) = u_0(r^{\frac{1}{2}}, \theta/2) \\ u_2(r, \theta) = u_0(r^{\frac{1}{2}}, \theta/2 + \pi) \end{cases}$$

for $0 < r < 1$ and $\theta_0 < \theta < \theta_0 + 2\pi$.

- $G \cap ((\mathcal{D} \setminus \{re^{i\theta_0}\}) \times \mathbb{R}) = G_1 \cup G_2$.

[SW07]: If $u_0 \notin C(\mathcal{D})$, then at a dense set of “good” heights $t_0 \in (\underline{\lim}_{r \rightarrow 0} u_0, \overline{\lim}_{r \rightarrow 0} u_0)$

$$G \cap B_\rho((0, t_0)) =$$

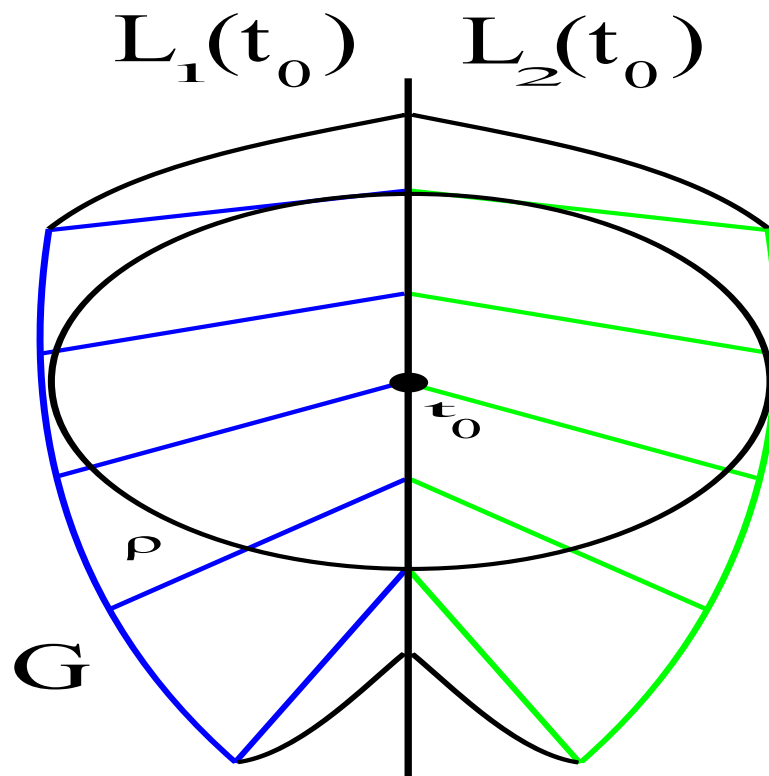


increasing/decreasing

If $\varphi_0(\theta + \frac{2\pi}{k}) = \varphi_0(\theta)$:

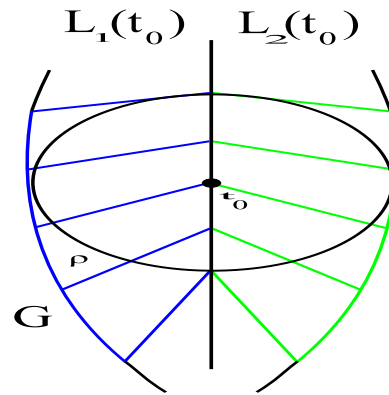
- $u_0(r, \theta + \frac{2\pi}{k}) = u_0(r, \theta)$, and G is $\mathbb{Z}_{\frac{2\pi}{k}}$.

- But if $u_0 \notin C(\mathcal{D})$:

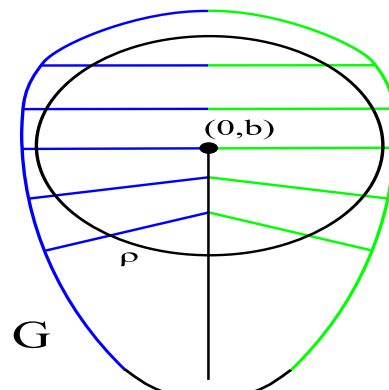


II: A More Complete Description

[R07]: If $u_0 \notin C(\mathcal{D})$, then every $t_0 \in (\underline{\lim}_{r \rightarrow 0} u_0, \overline{\lim}_{r \rightarrow 0} u_0)$ is good:



At the endpoints, G is a graph off a vertical plane:



Proof: Assume φ_0 even, let $u_0 \notin C(\mathcal{D})$.

- $u_0(r, -\theta) = u_0(r, \theta)$.

- Cut G along $x_1 > 0$:

$$\begin{cases} u_1(r, \theta) = u_0(r^{\frac{1}{2}}, \theta/2) \\ u_2(r, \theta) = u_0(r^{\frac{1}{2}}, \theta/2 + \pi) \end{cases}$$

for $0 < r < 1$ and $0 < \theta < 2\pi$.

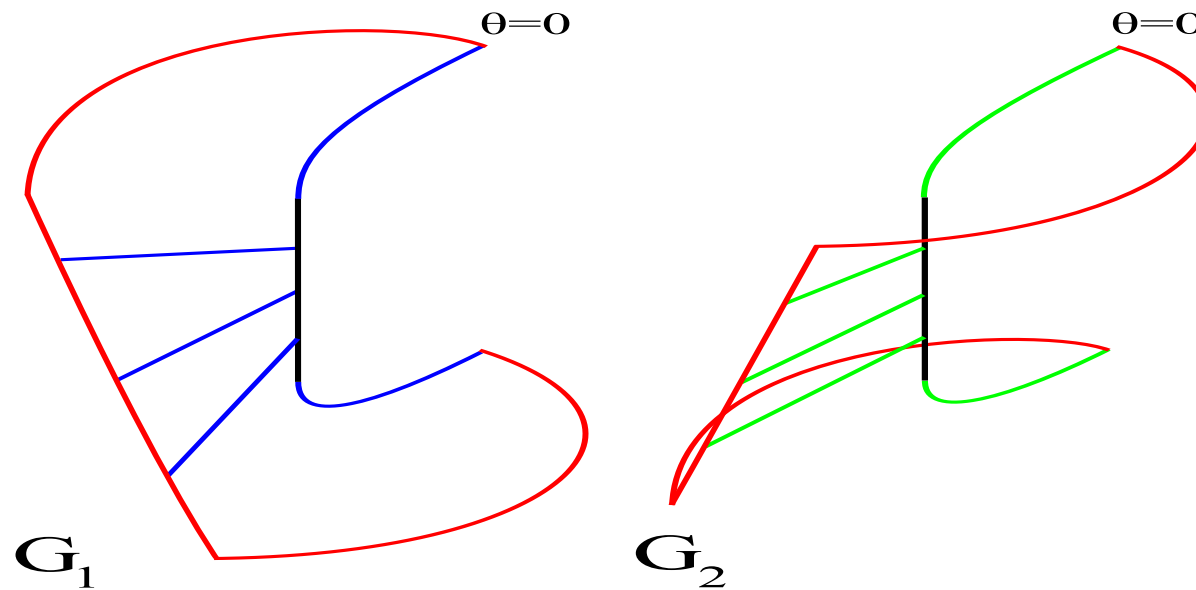
- Then $G_1 =$ reflection across x_1 of G_2 :

$$u_1(r, 2\pi - \theta) = u_2(r, \theta).$$

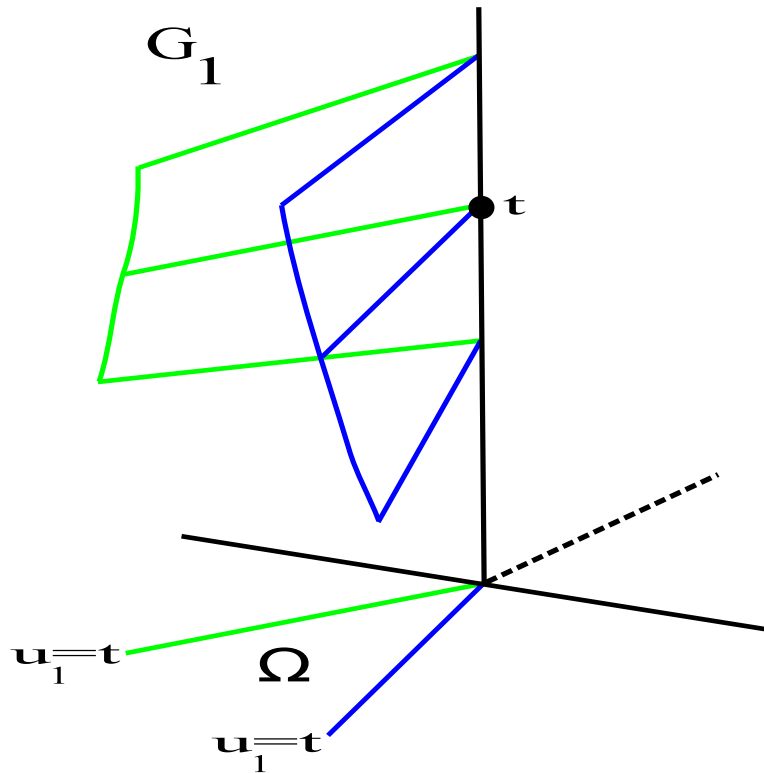
Lemma: If t is a good height (dense)

$$L_1(t) \subset G_1 \cap \{x_1 < 0\}$$

$$L_2(t) \subset G_2 \cap \{x_1 < 0\}.$$



Proof: Suppose $L_1(t), L_2(t) \subset G_1 \cap \{x_1 < 0\}$

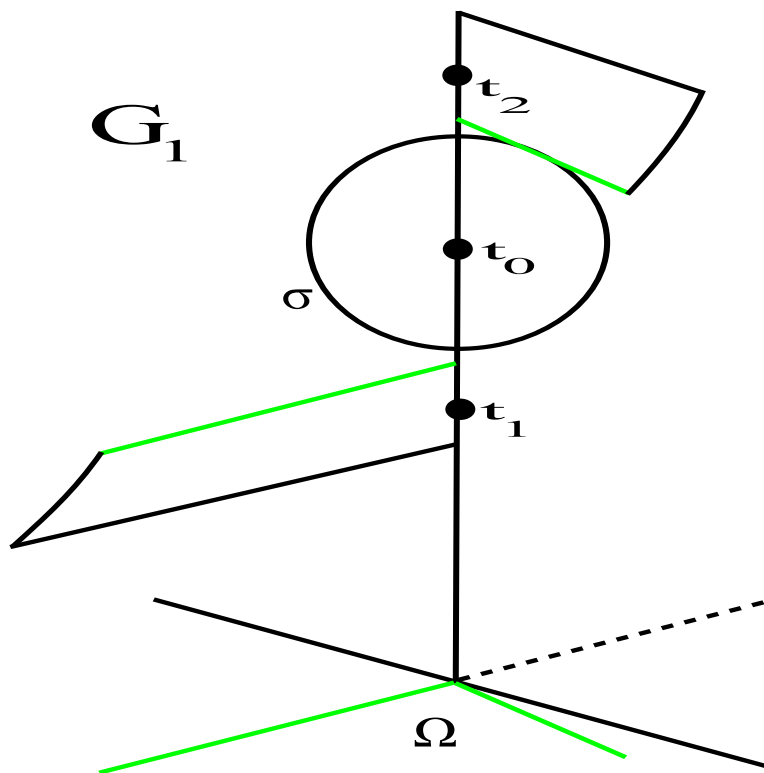


Since $\Omega \subset \{x_1 < 0\}$, a barrier argument gives contradiction:

$$\lim_{x \rightarrow 0, x \in \Omega} u_1(x) = t.$$

Proof: All heights are good.

Let $t_0 \in (\underline{\lim}_{r \rightarrow 0} u_0, \overline{\lim}_{r \rightarrow 0} u_0)$, choose good heights $t_1 < t_0 < t_2$.



We now apply the Theory of Varifolds:

- $\partial G_1 \cap B_\sigma((0, t_0)) = \{0\} \times (t_0 - \sigma, t_0 + \sigma)$:

$$G_1 + R_{\#}G_1$$

is stationary in $B_\sigma((0, t_0))$, by the Reflection Principle of [A75].

- $G_1 \cap B_\sigma((0, t_0)) = \text{graph}_\Omega u_1$ implies the tangent cones of G_1 at $(0, t_0)$ are sums of vertical half-planes:

$$\mathbb{C} = \sum_{k=1}^{q_0} |\mathcal{H}_k|, \text{ with } \mathcal{H}_k \subset \{x_1 < 0\}.$$

By a compactness result of [SS81] for stable surfaces.

- If $\mathbb{C} = \sum_{k=1}^{q_0} |\mathcal{H}_k|$, then for σ small

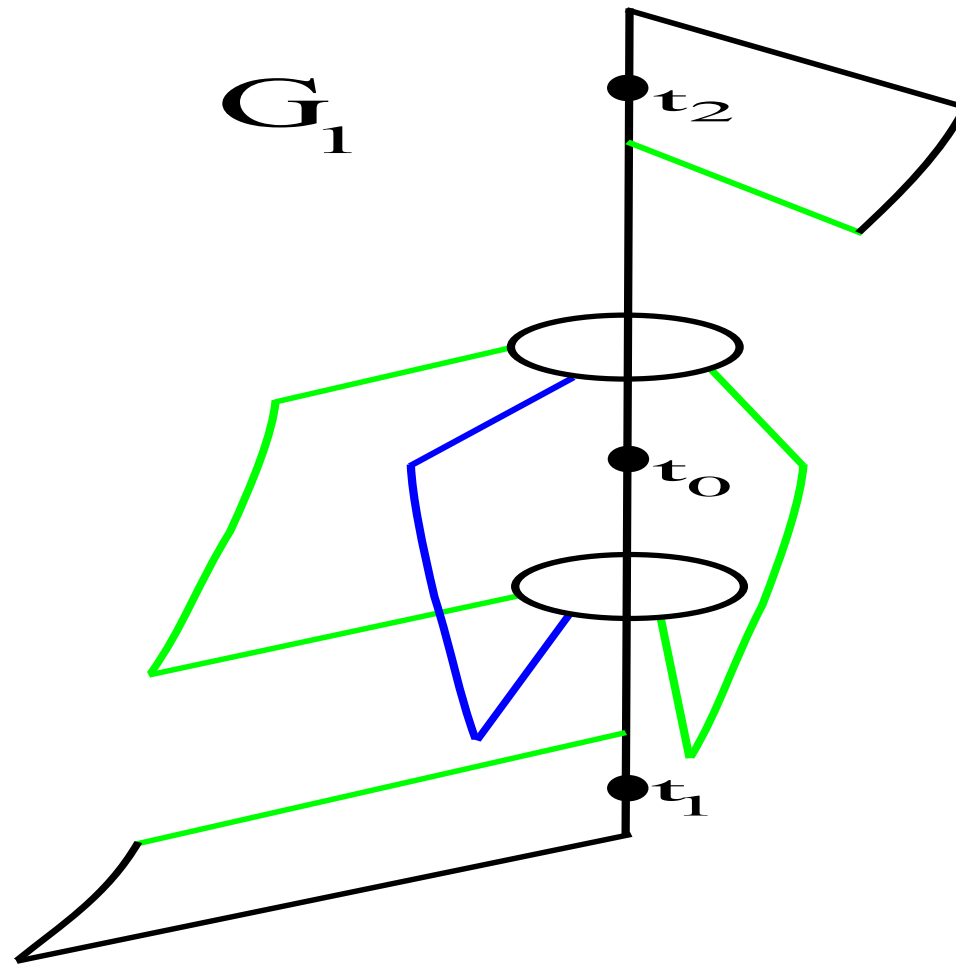
$$\text{dist}_{\mathcal{H}}(G_1 \cap B_\sigma((0, t_0)), \bigcup_{k=1}^{q_0} \mathcal{H}_k) < \delta\sigma.$$

- [SS81] states “a stable surface close to a plane is a graph over that plane,” and so:

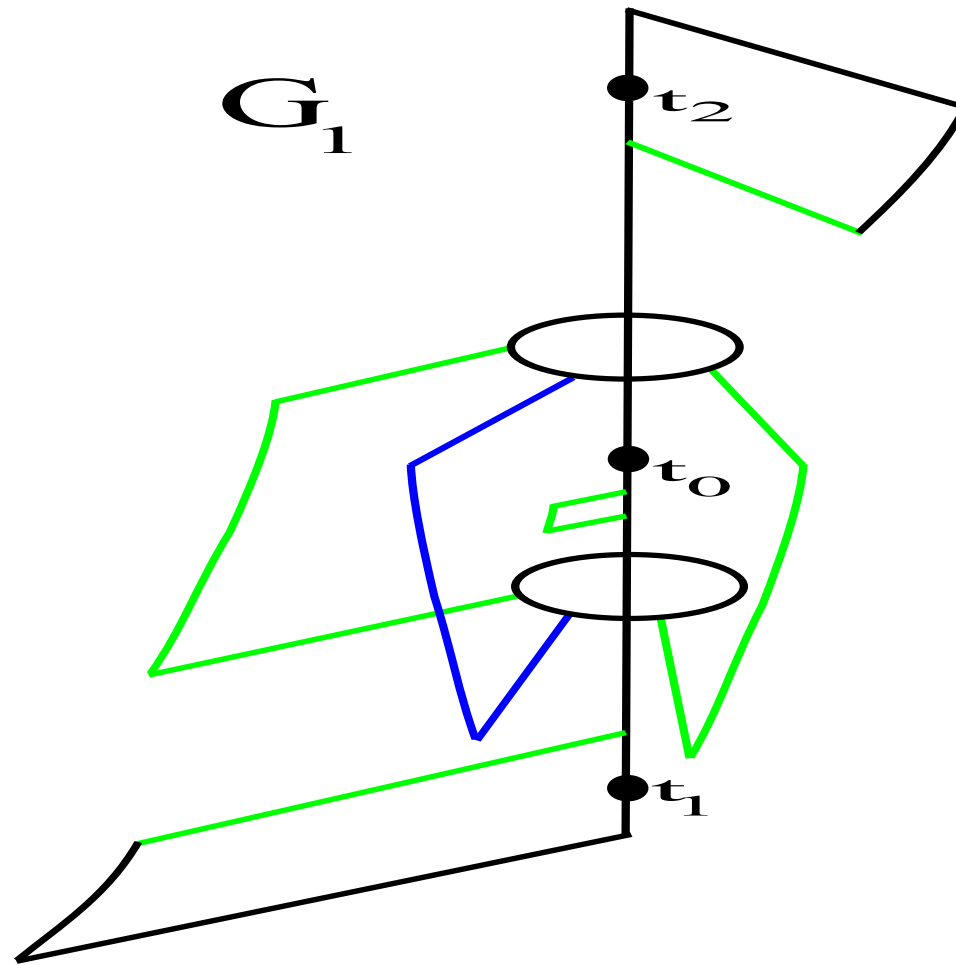
$$G_1 \cap B_\sigma((0, t_0)) \setminus (\mathcal{D}_{\delta\sigma} \times \mathbb{R}) = \bigcup_{k=1}^{q_0} \mathcal{L}_k$$

with

$$\mathcal{L}_k = \text{graph over } \mathcal{H}_k.$$

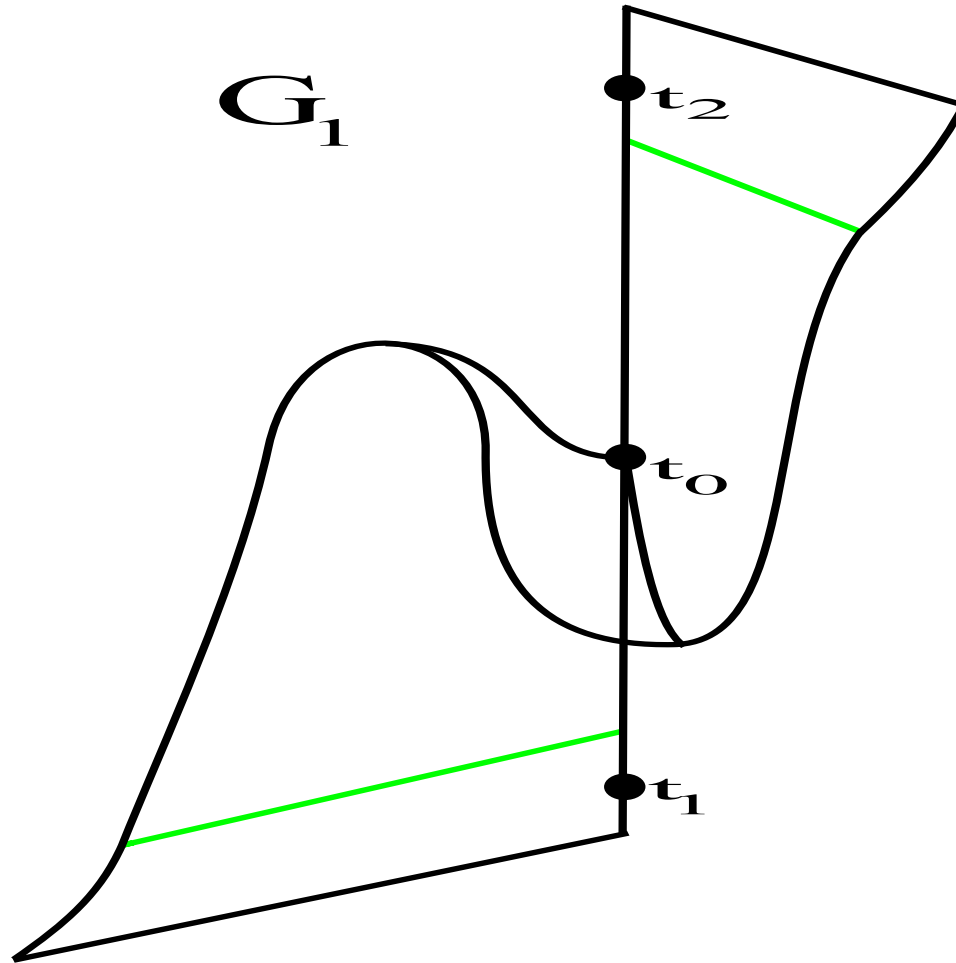


- By Allard's Boundary Regularity Thm [A75], suppose $q_0 = 3$.



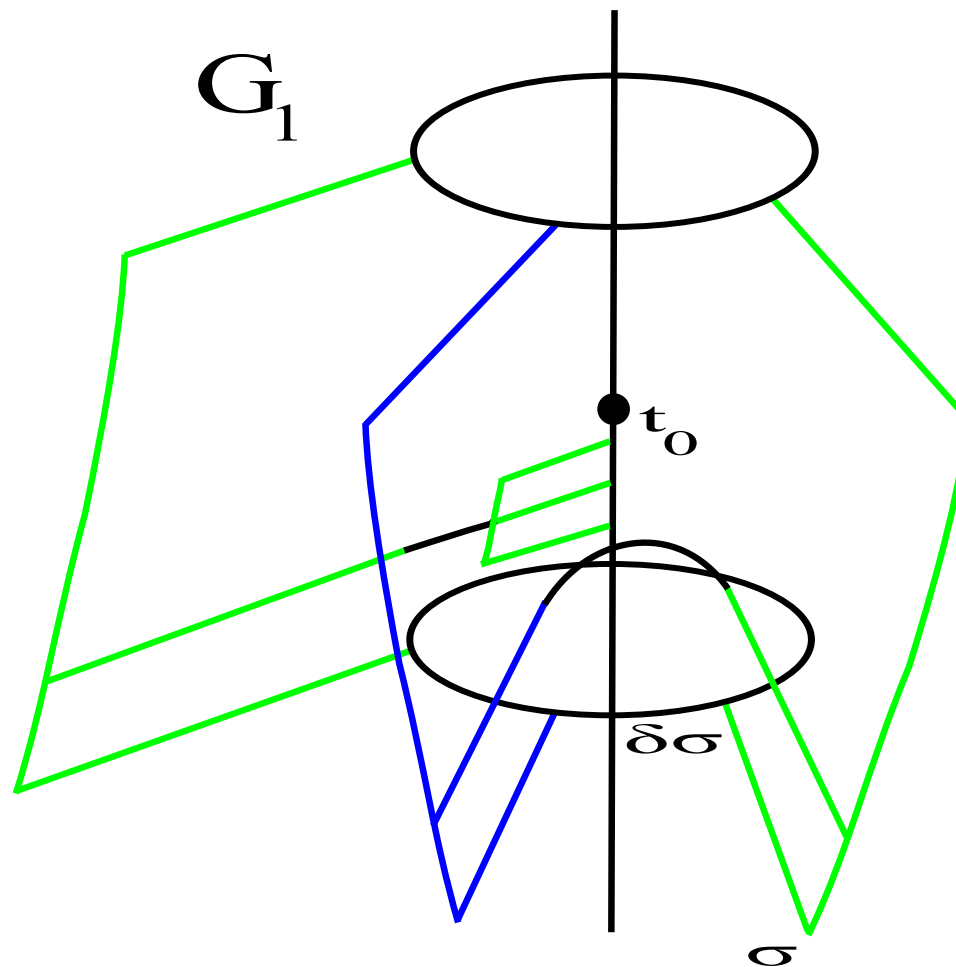
- Suppose $t \in (t_0 - \sigma, t_0)$ are good heights.

The problem we want to rule out:

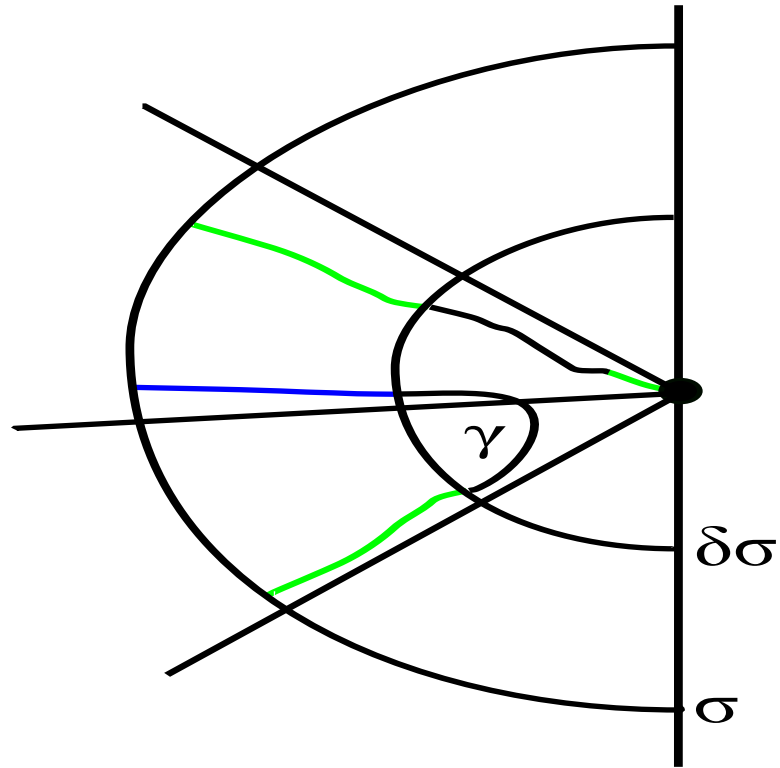


- Applying Sard's Thm for $t_0 - \sigma < t < t_0$, then

$$G_1 \cap (\mathcal{D}_\sigma \times \{t\}) =$$

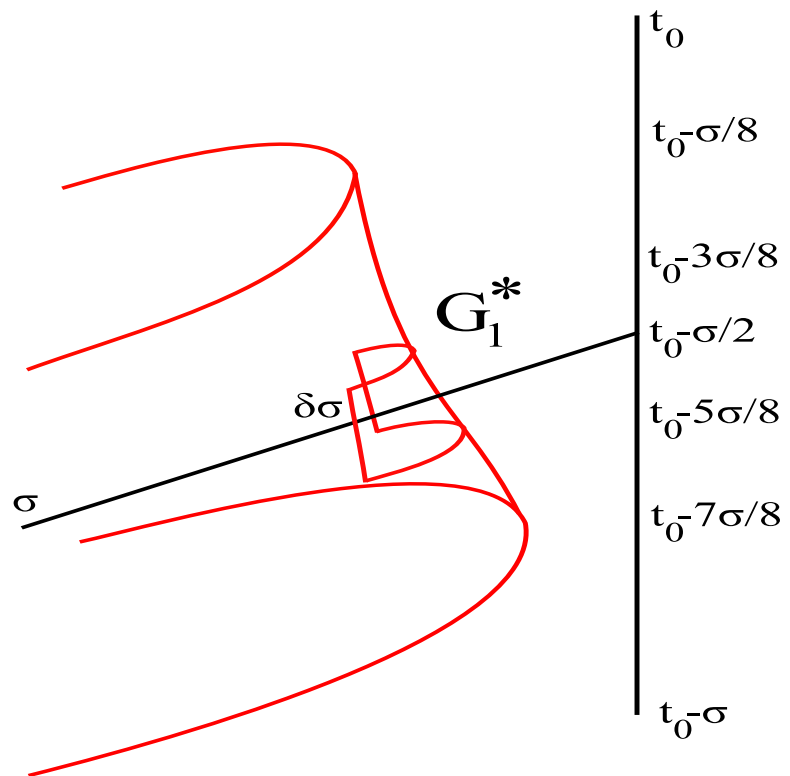


- Consider the cross-sections:



$$c(\pi - \vartheta) \leq |\nu_{G_1}(\gamma(1)) - \nu_{G_1}(\gamma(0))| \leq \int_{\gamma} |A_{G_1}| d\mathcal{H}^1$$

- We get $G_1^* \subset \mathcal{D}_{\delta\sigma} \times [t_0 - \frac{5\sigma}{8}, t_0 - \frac{3\sigma}{8}]$



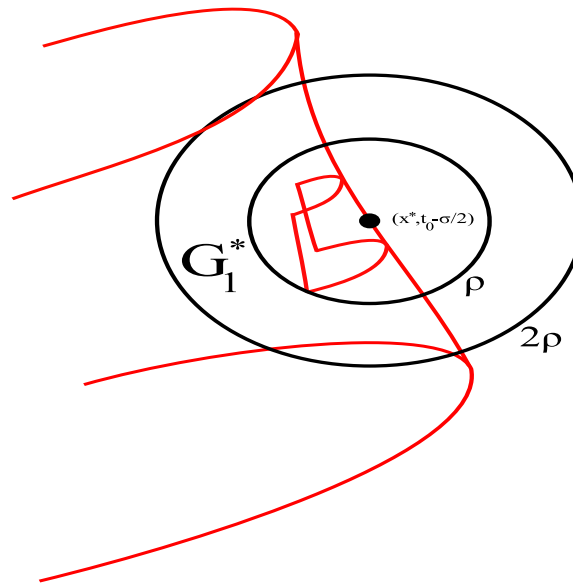
$$c(\pi - \vartheta)\sigma \leq \int_{G_1^*} |A_{G_1}| d\mathcal{H}^2$$

[SS83]: $M \subset B_1(0)$ simply connected, minimal, with $0 \in M$ and $B_{2\rho}(0) \subset B_1(0)$:

$$\int_{M_\rho^*} |A_M|^2 \leq c_{13} \left(\frac{\mathcal{H}^2(M \cap B_{2\rho}(0))}{(2\rho)^2} + 1 \right)$$

M_ρ^* = component of $M \cap B_\rho(0)$ with $0 \in M_\rho^*$.

- $\rho = \frac{3\sigma}{16}$



- However,

$$c_{13} \left(\frac{\mathcal{H}^2(G_1 \cap B_{2\rho}(x^*, t_0 - \sigma/2))}{(2\rho)^2} + 1 \right) \leq \text{Uniform Mass Bound.}$$

- Using Cauchy-Schwartz:

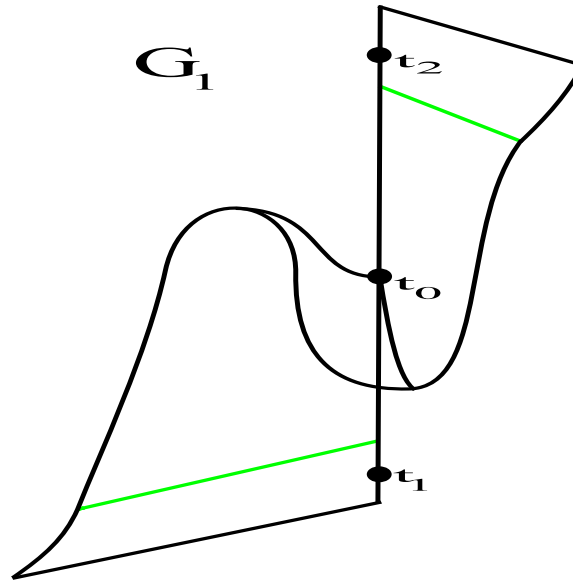
$$c(\pi - \vartheta)\sigma \leq \int_{G_1^*} |A_{G_1}| d\mathcal{H}^2 \leq \left(\int_{G_1^*} |A_{G_1}|^2 d\mathcal{H}^2 \right)^{1/2} \cdot \left(\int_{G_1^*} d\mathcal{H}^2 \right)^{1/2}$$

- So by [SS83]:

$$(\pi - \vartheta)\sigma \leq c \cdot \left(\int_{G_1^*} d\mathcal{H}^2 \right)^{1/2} \leq c \left(\frac{(\delta\sigma)^2}{\sigma} \right)^{1/2} = c\delta\sigma$$

since $G_1^* \subset \mathcal{D}_{\delta\sigma} \times [t_0 - \frac{5\sigma}{8}, t_0 - \frac{3\sigma}{8}]$.

Thus, we cannot have:



Thm: If $u_0 \notin C(\mathcal{D})$, because the set of good heights is dense,

$$G \cap B_\rho((0, t)) = L_1(t) \cup L_2(t)$$

for every $t \in (\underline{\lim}_{r \rightarrow 0} u_0, \overline{\lim}_{r \rightarrow 0} u_0)$.

- A similar analysis shows G is a graph off a vertical plane near the endpoints. For φ_0 even, the vertical plane is $\{x_1 = 0\}$.
- [SW07]: There is a dense set of good heights.

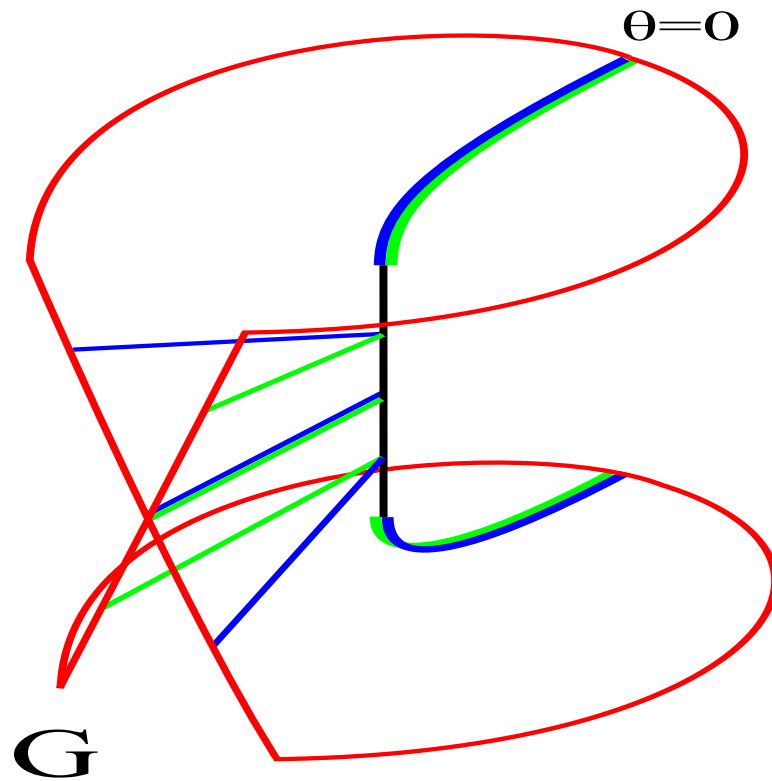
Proof: Consider points of almost local minimum density.

Let $\kappa = \inf_{t \in [a, b]} \Theta_G((0, t))$, then for all ϵ , $\exists t_0, \rho_0$, so that for all $t \in (t_0 - \rho_0, t_0 + \rho_0)$, $\rho \leq \rho_0$

$$\kappa \leq \frac{\mathcal{H}^2(G \cap B_\rho((0, t)))}{\pi \rho^2} < \kappa + \epsilon.$$

III: Discontinuous Solutions

[R07]: For $\alpha \neq 0$, the data $\varphi_0(\theta) = \alpha \cos(\theta)$, gives $u_0 \notin C(\mathcal{D})$.

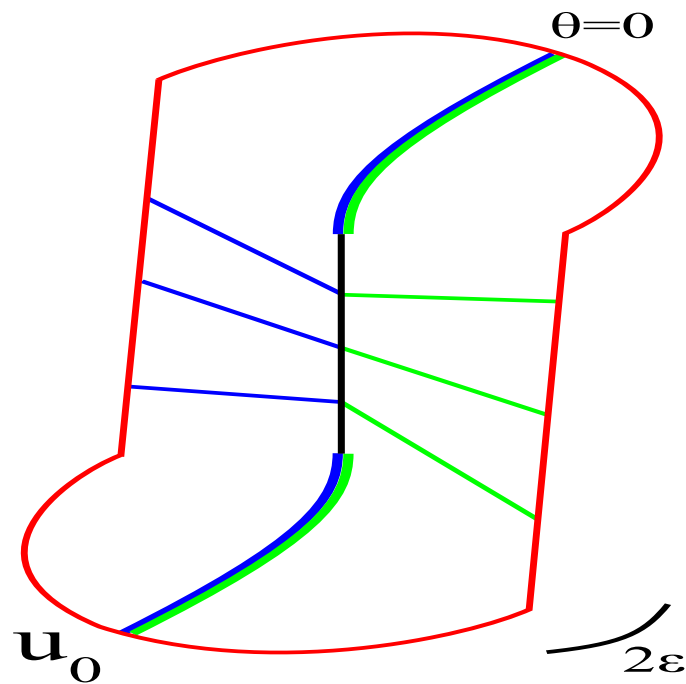


Thm: For all $M > 0$ there exists ϵ so that if $\varphi_0 \in C(S^1)$ is even with

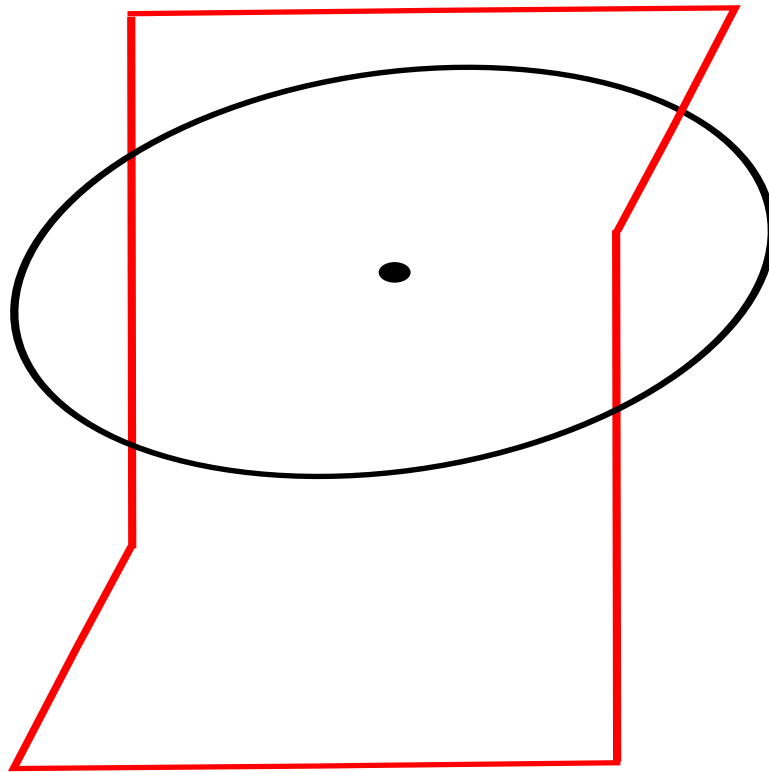
$$M < \varphi_0 \quad \text{for } \theta \in (0, \pi/2 - \epsilon)$$

$$\varphi_0 < -M \quad \text{for } \theta \in (\pi/2 + \epsilon, \pi)$$

and $\|\varphi_0\|_{S^1} < M + \epsilon$, then $u_0 \notin C(\mathcal{D})$.

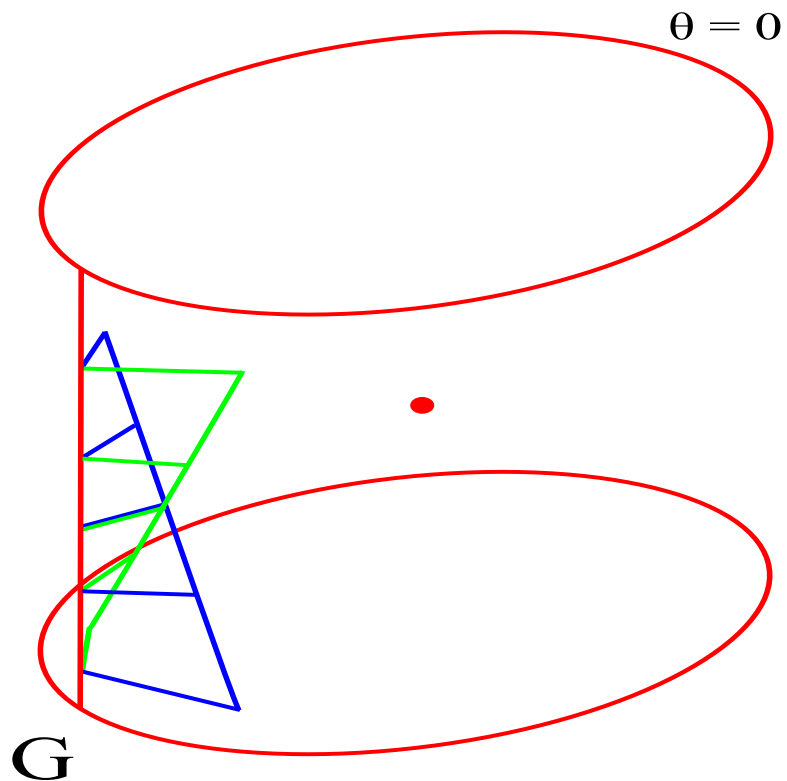


[F63]: Gradient estimates for solutions to the MSE, through the construction of comparison surfaces.

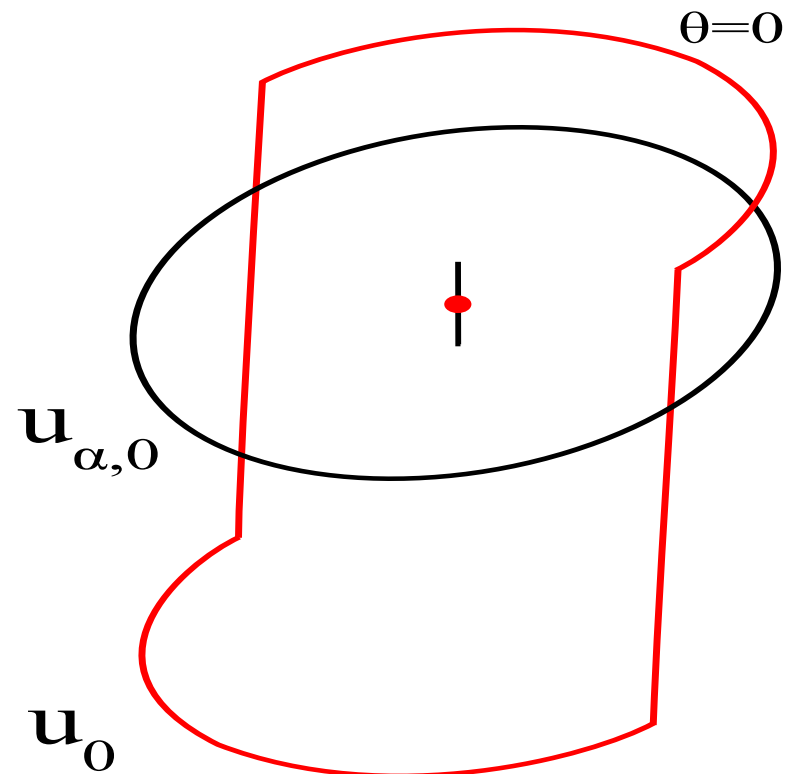


Proof: Let $\varphi_0^j \rightarrow M\mathcal{X}_{(0,\pi/2)} - M\mathcal{X}_{(\pi/2,\pi)}$ with $u_0^j \in C(\mathcal{D})$.

- Interior gradient estimates gives $u_0^j \rightarrow u_0 \in C(\mathcal{D})$.
- Maximum Principle: $-M < u_0(0) < M$.

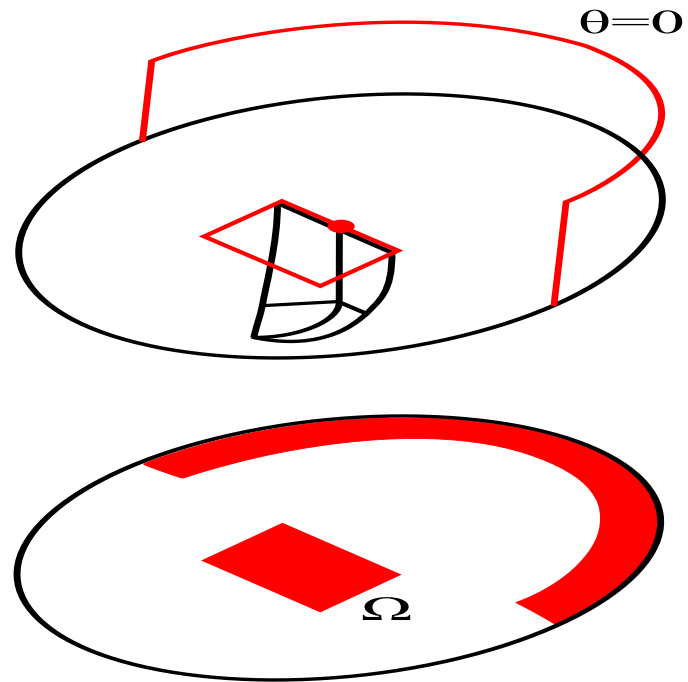


- Choose $\rho \rightarrow 1$ so that $u_0(\rho, \cdot)$ is two steep lines at height $u_0(0)$:



- Consider $u_{\alpha,0}$ with $\varphi_{\alpha,0}(\theta) = \alpha \cos(\theta) + u_0(0)$ on \mathcal{D}_ρ .

- Consider $\{x \in \mathcal{D}_\rho : u_0 - u_{\alpha,0} > 0\}$:



then $\partial\Omega \not\subseteq \{u_0 - u_{\alpha,0} = 0\}$, by the Maximum Principle.

- Considering $\{u_0 - u_{\alpha,0} < 0\}$, we get a contradiction.

We can also consider q -Valued MSE

$$\mathcal{M}_0(v) = \sum_{j=1}^2 D_{x_j} \left(\frac{D_{x_j} v}{1 + \frac{|Dv|^2}{(qr^{q-1})^2}} \right)$$

- $u_0 \notin C(\mathcal{D})$, then for all but finitely many $t \in (\underline{\lim}_{r \rightarrow 0} u_0, \overline{\lim}_{r \rightarrow 0} u_0)$

$$G \cap B_\rho((0, t)) = \bigcup_{j=1}^{q_0} L_j(t)$$

$2 \leq q_0 \leq 2q - 2$ even.

- For $q = 3$, $\cos(2\theta)$ gives $u_0 \notin C(\mathcal{D})$.

