

**Second Midterm Practice Exam**  
**Math 355**  
**Summer 2008**

**Instructions:** This is not a timed exam, however you must complete the exam all at once. You may not consult any notes or books during the exam, and no calculators are allowed. You may not discuss this exam with anyone except your instructor. Show all of your work on each problem.

1. Determine whether the following statements are true or false. If true, briefly explain why. If false, provide a counterexample.

(a) The norm induced by an inner product is  $\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x} \rangle$ .

False. It is  $\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$ .

(b) An  $n \times n$  matrix  $A$  is diagonalizable if it has  $n$  distinct eigenvalues.

True. If it has  $n$  distinct eigenvalues, then it has  $n$  linearly independent eigenvectors, and thus it is diagonalizable.

(c) Every normed linear space is also an inner product space.

False. A normed linear space is an inner product space iff the norm satisfies the parallelogram law.  $\mathbb{R}^2$  with the norm  $\|\mathbf{x}\|_p = (|x_1|^p + |x_2|^p)^{1/p}$  with  $p \neq 2$  is not an inner product space, for example.

(d) If  $A$  is an  $n \times n$  matrix and has 0 as an eigenvalue, then  $A$  is singular.

True.  $A$  is nonsingular iff it doesn't have an eigenvalue of 0.

(e) If  $A^\perp = B$ , then  $B^\perp = A$ .

True.  $(S^\perp)^\perp = S$ , so  $A^\perp = B$  implies  $(A^\perp)^\perp = B^\perp$  implies  $A = B^\perp$ .

(f) A linear transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  can be 1-1.

True. It is possible (not always true, but possible) for a linear transformation  $T : V \rightarrow W$  to be 1-1 as long as  $\dim V \leq \dim W$ . Here, for example, just take  $T(\mathbf{x}) = (x_1, x_2, 0)^T$  and only  $(0, 0)^T$  gets sent to  $(0, 0, 0)^T$  by  $T$ .

(g) An  $n \times n$  real valued matrix can have a single complex eigenvalue.

False. Complex eigenvalues always come in pairs.

2. Consider the matrix  $A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 4 & 0 \end{pmatrix}$ .

- (a) Find  $R(A^T)$  and  $N(A)$ .

$A^T$  row reduces to  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ , so  $R(A^T) = \text{Span} \left( \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} \right)$ .

Also,  $A$  reduces to  $\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$ , so  $x_3$  is free and a basis for  $N(A)$  is  $\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\}$ .

- (b) What is the relationship between  $R(A^T)$  and  $N(A)$ ?

$R(A^T)^\perp = N(A)$ , which is easily verified from the bases found above for the two subspaces. The fact that they are orthogonal complements and not just orthogonal subspaces is seen because  $\dim N(A) + \dim R(A^T) = \dim \mathbb{R}^3$ .

3. Let  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}^3$ . If  $\mathbf{x}_1 \perp \mathbf{x}_2$  and  $\mathbf{x}_2 \perp \mathbf{x}_3$ , must  $\mathbf{x}_1 \perp \mathbf{x}_3$ ?

No. For example, it may be true that  $x_1 = x_3$ . Or that  $x_1 = -x_3$ , etc.

4. Let  $T : P_2 \rightarrow P_3$  be the linear transformation given by

$$T(p(x)) = x^2 \cdot p'(x).$$

- (a) Find a basis for the kernel of  $T$ .

$T(ax + b) = x^2(a) = ax^2$ , so if this equals 0, then  $a = 0$ . Thus all vectors in  $P_2$  with  $a = 0$ ; i.e. all vectors of the form  $b$ , get sent to 0 under  $T$ . So  $\{1\}$  is a basis for the kernel of  $T$ .

- (b) Find the matrix representation of  $T$  with respect to the standard basis  $\mathcal{E}_3 = \{1, x, x^2\}$  for  $P_3$  and  $\mathcal{E}_2 = \{1, x\}$  for  $P_2$ .

To do this, we take  $T(1)$  and this becomes the first column of  $A$ , and  $T(x)$  is the second column, all wrt  $\mathcal{E}_3$ .  $T(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$  and  $T(x) = x^2 = 0 \cdot 1 + 0 \cdot x + 1 \cdot x^2$ , so  $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$  is the matrix representation.

- (c) Consider the basis  $\mathcal{B} = \{x, x^2 + 1, 1\}$  for  $P_3$  and  $\mathcal{C} = \{1, 2 + x\}$  for  $P_2$ . Find the transition matrices from  $\mathcal{E}_3$  to  $\mathcal{B}$  and from  $\mathcal{E}_2$  to  $\mathcal{C}$ , and then find the matrix that represents  $T$  with respect to  $\mathcal{C}$  and  $\mathcal{B}$ .

First, we want to find the transition matrix that takes  $\mathcal{C}$  to the standard basis of  $P_2$ . To do this, simply form the matrix with 1 and  $2 + x$  as the columns:  $S = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ . Similarly, the transition matrix from  $\mathcal{B}$  to  $\mathcal{E}_3$  is  $V = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ . So the matrix that represents  $T$  with respect

to  $\mathcal{C}$  and  $\mathcal{B}$  is  $V^{-1}AS$ , because first we have to take  $\mathcal{C}$  to the standard basis by  $S$ , then apply the transformation  $A$ , and then convert from the

standard basis of  $P_3$  to  $\mathcal{B}$  by  $V^{-1}$ .  $V^{-1}AS = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}$ , as  $V^{-1} =$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

- (d) To check your answer, consider  $3x + 2 \in P_2$ . Find its coordinates wrt  $\mathcal{C}$  and then apply the matrix from (c) to that coordinate vector. What you get out is a coordinate vector wrt  $\mathcal{B}$ . Show that this is equal to  $T(3x+2)$ .

$3x + 2 = -4 \cdot 1 + 3(2 + x)$ , so the coordinate vector wrt  $\mathcal{C}$  is  $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$ , and

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix}.$$

This should be the coordinate vector of  $3x + 2$  after the transformation  $T$  wrt  $\mathcal{B}$ , or  $3(x^2 + 1) - 3(1) = 3x^2$ . Sure enough,  $T(3x + 2) = 3x^2$ .

5. Use the Gram-Schmidt Orthogonalization Process to build up the following set into an orthonormal basis for  $\mathbb{R}^3$ :

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right\}.$$

First, we normalize  $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$  to get  $\mathbf{u}_1 = \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix}$ . Then, projecting  $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

onto  $\mathbf{u}_1$  gives  $\mathbf{p}_2 = \begin{pmatrix} 8/9 \\ 4/9 \\ 8/9 \end{pmatrix}$ , and  $\mathbf{u}_2 = \frac{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 8/9 \\ 4/9 \\ 8/9 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 8/9 \\ 4/9 \\ 8/9 \end{pmatrix} \right\|} = \frac{3}{\sqrt{29}} \begin{pmatrix} 1/9 \\ 14/9 \\ -8/9 \end{pmatrix} =$

$\begin{pmatrix} 1/3\sqrt{29} \\ 14/3\sqrt{29} \\ -8/3\sqrt{29} \end{pmatrix}$ . Finally, we need to add another linearly independent vector

to make a basis, and then make it orthogonal to the other two using Gram-Schmidt. To simplify, let's choose one that is already orthogonal to  $\mathbf{u}_2$ . Consider  $\begin{pmatrix} -14 \\ 1 \\ 0 \end{pmatrix}$ , which is orthogonal to  $\mathbf{u}_2$  and not a multiple of  $\mathbf{u}_1$ , so it is

independent. Now projection onto the  $\text{Span}(\mathbf{u}_1, \mathbf{u}_2)$ , we get  $\mathbf{p}_3 = \begin{pmatrix} -6 \\ -3 \\ -6 \end{pmatrix}$

$$\text{and } \mathbf{u}_3 = \frac{\begin{pmatrix} -14 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -6 \\ -3 \\ -6 \end{pmatrix}}{\left\| \begin{pmatrix} -14 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -6 \\ -3 \\ -6 \end{pmatrix} \right\|} = 2\sqrt{29} \begin{pmatrix} -8 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} -4/\sqrt{29} \\ 2/\sqrt{29} \\ 3/\sqrt{29} \end{pmatrix}. \text{ And}$$

(you can check)  $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$  forms an orthonormal basis for  $\mathbb{R}^3$  from the original vectors.

6. Consider the matrix  $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{pmatrix}$ .

- (a) Find all of the eigenvalues of  $A$  and the corresponding eigenvectors/eigenspaces.

For a diagonal matrix, the eigenvalues are just along the diagonal. So the eigenvalues are 1 (repeated) and 3. Considering  $N(A - I)$ , we find that the eigenvectors corresponding to 1 are nonzero multiples of  $(1, 0, 0)^T$ . And considering  $N(A - 3I)$ , we find that the eigenvectors corresponding to 3 are nonzero multiples of  $(2, -1/2, 1)^T$ .

- (b) Is  $A$  diagonalizable? If so, factor  $A$  into a product  $DXD^{-1}$ , where  $D$  is diagonal. If not, explain why. It is not, because an  $n \times n$  matrix is diagonalizable iff it has  $n$  linearly independent eigenvectors, and this matrix only has 2 linearly independent eigenvectors. The problem is that the eigenspace corresponding to 1 is only of dimension 1, instead of dimension 2.

7. Prove that a linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is 1-1 if and only if  $T(\mathbf{x}) = \mathbf{0}$  implies  $\mathbf{x} = \mathbf{0}$ .

Suppose  $T$  is 1-1. Then if  $T(\mathbf{x}_1) = T(\mathbf{x}_2)$ , we know  $\mathbf{x}_1 = \mathbf{x}_2$ . Let  $T(\mathbf{x}) = \mathbf{0}$ . Then  $T(\mathbf{x}) = T(\mathbf{0})$ , but  $T$  is 1-1, so this implies that  $\mathbf{x} = \mathbf{0}$ , as desired.

Now suppose that  $T(\mathbf{x}) = \mathbf{0}$  implies  $\mathbf{x} = \mathbf{0}$ . If  $T(\mathbf{x}_1) = T(\mathbf{x}_2)$ , then  $T(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{0}$ , but by assumption this implies that  $\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{0}$ , or that  $\mathbf{x}_1 = \mathbf{x}_2$ , as desired.

8. Let  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  and  $T^* : \mathbb{R}^3 \rightarrow \mathbb{R}^4$  be given by

$$T((x_1, x_2, x_3, x_4)^T) = (x_1 + x_2, x_3, 4x_1 + x_2 + x_3)^T,$$

$$T^*((x_1, x_2, x_3)^T) = (x_1 + 4x_3, x_1 - 2x_3, x_3, x_1)^T.$$

- (a) Prove that  $T$  and  $T^*$  are linear transformations.

Show that  $T(\alpha\mathbf{x}) = \alpha T(\mathbf{x})$  and that  $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ .

- (b) Find the matrix representations for  $T$  and  $T^*$  with respect to the standard basis.  $T(\mathbf{e}_1) = (1, 0, 4)^T, T(\mathbf{e}_2) = (1, 0, 1)^T, T(\mathbf{e}_3) = (0, 1, 1)^T,$

and  $T(\mathbf{e}_4) = (0, 0, 0)^T$ , so  $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 1 & 1 & 0 \end{pmatrix}$ . Similarly,  $A^* =$

$$\begin{pmatrix} 1 & 0 & 4 \\ 1 & 0 & 2 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{pmatrix}.$$

- (c) Are  $T$  and  $T^*$  1-1? Are they onto? Explain. We can answer this by considering the matrices  $A$  and  $A^*$ . The nullspace of  $A$ ,  $N(A)$ , is nontrivial, because, for example, the fourth column can never have a pivot, so the transformation  $T$  cannot be 1-1 because the kernel is nontrivial. The columns of  $A$  span  $\mathbb{R}^3$ , however, because the first three are linearly independent. Thus  $T$  maps  $\mathbb{R}^4$  onto  $\mathbb{R}^3$ .

The nullspace of  $A^*$  is similarly nontrivial, so  $T^*$  is not 1-1. It is also not onto, because there is no way for three column vectors in  $\mathbb{R}^4$  to span the entire space.