

First Midterm Practice Exam
Math 355
Summer 2008

Instructions: This is not a timed exam, however you must complete the exam all at once. You may not consult any notes or books during the exam, and no calculators are allowed. You may not discuss this exam with anyone except your instructor. Show all of your work on each problem.

1. Determine whether the following statements are true or false. If true, briefly explain why. If false, provide a counterexample.

- (a) A homogeneous linear system is always consistent.

True. $\mathbf{0}$ is always a solution to $A\mathbf{x} = \mathbf{0}$.

- (b) An $n \times n$ matrix is nonsingular if and only if its diagonal entries are all nonzero.

False. Consider $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. It could be changed into one of the following true statements:

- i. An $n \times n$ triangular matrix is nonsingular if and only if its diagonal entries are all nonzero.
- ii. An $n \times n$ matrix in row echelon form is nonsingular if and only if its diagonal entries are all nonzero.

- (c) If A and B are nonsingular $n \times n$ matrices, then $A + B$ is also nonsingular and $(A + B)^{-1} = A^{-1} + B^{-1}$.

False. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

- (d) If $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ span \mathbb{R}^n , then they are linearly independent.

True. If a set of n vectors span \mathbb{R}^n , then they form a basis and are therefore linearly independent.

- (e) If A is an $m \times n$ matrix, then the rank of A plus the nullity of A equals n .

True. This is the Rank-Nullity Theorem.

- (f) If U and V are subspaces of \mathbb{R}^n and $U \cap V = \{\mathbf{0}\}$, then $\dim(U + V) = \dim U + \dim V$.

True. If one combines a basis for U and a basis for V , one gets a basis for $U + V$. This requires a little work to show rigorously.

2. Consider the following matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$.

- (a) Find A^{-1} .

$$A^{-1} = \begin{pmatrix} 1/7 & -2/7 & 3/7 \\ 4/7 & -1/7 & -2/7 \\ -2/7 & 4/7 & 1/7 \end{pmatrix}.$$

- (b) Use part (a) to solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

$$\begin{pmatrix} 1/7 & -2/7 & 3/7 \\ 4/7 & -1/7 & -2/7 \\ -2/7 & 4/7 & 1/7 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6/7 \\ -4/7 \\ 9/7 \end{pmatrix} \text{ is the solution to } A\mathbf{x} = \mathbf{b}.$$

3. Let A be an 5×4 matrix with rank 3. Is $N(A)$ trivial or nontrivial? Explain. Hint: Use the Rank-Nullity Theorem.

By the Rank-Nullity Theorem, the rank of A plus the nullity of A equals 4. So 3 plus the nullity equals 4, or the nullity equals 1. That is, $\dim N(A) = 1$, and the nullspace of A is nontrivial.

4. Consider the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}.$$

- (a) Is the set linearly independent or linearly dependent?

$$\begin{vmatrix} 2 & 3 & 2 \\ 1 & -1 & 6 \\ 3 & 4 & 4 \end{vmatrix} = 0, \text{ so the set is dependent.}$$

- (b) What is the dimension of $\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$?

In this case, because the vectors are pairwise linearly independent, only one is unnecessary - any two will form a basis for the span. Note that this is not always the case!! So the dimension is 2.

5. Consider the subspace S of $\mathbb{R}^{2 \times 2}$ consisting of all matrices of the form

$$\begin{pmatrix} a & 0 \\ a - c & a + c \end{pmatrix}.$$

(a) Is this a proper subspace? Explain.

Yes, it is proper. $S \neq \mathbb{R}^{2 \times 2}$, because, for example, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{2 \times 2}$ but it is not in S . Also, $S \neq \{O\}$, because, for example, $O \neq \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in S$.

(b) Find a basis for S .

$$\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} \right\}$$

6. Let S be the vector space of all infinite sequences of real numbers with scalar multiplication and addition defined by

$$\begin{aligned} \alpha\{a_n\} &= \{\alpha a_n\} \\ \{a_n\} + \{b_n\} &= \{a_n + b_n\}. \end{aligned}$$

Let S_0 be the set of $\{a_n\}$ with the property $a_n \rightarrow 0$ as $n \rightarrow \infty$. Show that S_0 is a subspace of S .

Let $\{a_n\} \in S_0$. As $\alpha\{a_n\} = \{\alpha a_n\}$, and $\lim_{n \rightarrow \infty} \alpha a_n = \alpha \lim_{n \rightarrow \infty} a_n = \alpha \cdot 0 = 0$, then $\alpha a_n \rightarrow 0$ as $n \rightarrow \infty$ and $\alpha\{a_n\} \in S_0$.

Also, let $\{b_n\} \in S_0$. As $\{a_n\} + \{b_n\} = \{a_n + b_n\}$, and $\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n = 0 + 0 = 0$, then $(a_n + b_n) \rightarrow 0$ as $n \rightarrow \infty$ and $\{a_n\} + \{b_n\} \in S_0$.

Finally, the sequence of all 0's is in S_0 , so it is nonempty. Thus S_0 is a subspace of S .

7. Prove that the multiplicative identity I for $n \times n$ matrices is unique; i.e., there is no other matrix J such that $JA = AJ = A$ for all $A \in \mathbb{R}^{n \times n}$.

Suppose there exists a matrix J such that $JA = AJ = A$ for all $A \in \mathbb{R}^{n \times n}$. Then $JJ = I$, as J is an identity, and also $JJ = J$, as I is an identity. So $J = I$, and the identity is unique.

8. Prove that a linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if the rank of $(A|\mathbf{b})$ equals the rank of A .

We know that $A\mathbf{x} = \mathbf{b}$ is consistent if and only if when the augmented matrix is in row echelon form, there is no row like $(00 \dots 0|c)$ where $c \neq 0$. If there is such a row, then the rank of $(A|\mathbf{b})$ is one greater than the rank of A . Similarly, if the rank of $(A|\mathbf{b})$ is different the rank of A , it must be greater, and the only way it can be greater is if there is a row like $(00 \dots 0|c)$ where $c \neq 0$ in the row reduced augmented matrix.

9. Suppose A is a nonsingular $n \times n$ matrix. List as many equivalent statements to this as you can recall.

(a) A is invertible.

(b) $\det(A) \neq 0$.

(c) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{0}$.

- (d) A is row equivalent to I .
- (e) $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b} \in \mathbb{R}^n$.
- (f) When A is reduced to row echelon form, there are no free columns/variables.
- (g) When A is reduced to row echelon form, there are nonzero entries along the diagonal.
- (h) $\text{Null}(A) = \mathbf{0}$.
- (i) The column vectors of A are a linearly independent set.
- (j) The column vectors of A form a basis for \mathbb{R}^n .