First Midterm Practice Exam
Math 355
Summer 2008

Instructions: This is not a timed exam, however you must complete the exam all at once. You may not consult any notes or books during the exam, and no calculators are allowed. You may not discuss this exam with anyone except your instructor. Show all of your work on each problem.

1. Determine whether the following statements are true or false. If true, briefly explain why. If false, provide a counterexample.
   (a) A homogeneous linear system is always consistent.
   (b) An $n \times n$ matrix is nonsingular if and only if its diagonal entries are all nonzero.
   (c) If A and B are nonsingular $n \times n$ matrices, then $A+B$ is also nonsingular and $(A+B)^{-1} = A^{-1} + B^{-1}$.
   (d) If $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$ span $\mathbb{R}^n$, then they are linearly independent.
   (e) If A is an $m \times n$ matrix, then the rank of A plus the nullity of A equals n.
   (f) If U and V are subspaces of $\mathbb{R}^n$ and $U \cap V = \{\mathbf{0}\}$, then $\dim(U+V) = \dim U + \dim V$.

2. Consider the following matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$.
   (a) Find A^{-1}.
   (b) Use part (a) to solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

3. Let A be an 5×4 matrix with rank 3. Is $N(A)$ trivial or nontrivial? Explain. Hint: Use the Rank-Nullity Theorem.

4. Consider the set of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$, where

$$\mathbf{v_1} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}, \mathbf{v_3} = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}.$$

   (a) Is the set linearly independent or linearly dependent?

   (b) What is the dimension of $\text{Span}(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})$?

5. Consider the subspace S of $\mathbb{R}^{2\times 2}$ consisting of all matrices of the form

$$\begin{pmatrix} a & 0 \\ a-c & a+c \end{pmatrix}.$$

   (a) Is this a proper subspace? Explain.

   (b) Find a basis for S.

6. Let S be the vector space of all infinite sequences of real numbers with scalar multiplication and addition defined by

$$\alpha\{a_n\} = \{\alpha a_n\}$$
$$\{a_n\} + \{b_n\} = \{a_n + b_n\}.$$

   Let S_0 be the set of $\{a_n\}$ with the property $a_n \to 0$ as $n \to \infty$. Show that S_0 is a subspace of S.

7. Prove that the multiplicative identity I for $n \times n$ matrices is unique; i.e., there is no other matrix J such that $JA = AJ = A$ for all $A \in \mathbb{R}^{n\times n}$.

8. Prove that a linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if the rank of $(A|\mathbf{b})$ equals the rank of A.

9. Suppose A is a nonsingular $n \times n$ matrix. List as many equivalent statements to this as you can recall.