6.3 GAUGE TRANSFORMATIONS

Results for scalar and vector potentials:

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} \tag{6.9}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{6.7}$$

Inhomogeneous Maxwell equations in terms of **A**, **Φ**

$$\nabla^2\Phi + \frac{\partial(\nabla\cdot\mathbf{A})}{\partial t} = -\rho/\varepsilon_o \tag{6.10}$$

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla\left[\nabla\cdot\boldsymbol{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right] = -\mu_o \boldsymbol{J} \tag{6.11}$$

Gauge transformations

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla\Lambda \tag{6.12}$$

$$\Phi \rightarrow \Phi - \frac{\partial\Lambda}{\partial t} \tag{6.13}$$

do not change **E** or **B**.

Conclusion: **A** and **Φ** can't be uniquely calculated from the observables **E** and **B**. To make them uniquely calculable, you have to install another condition—e.g., a differential equation. One usually chooses a condition, called a "gauge condition" that makes (6.10) and(6.11) look better.

Choice 1. Lorentz gauge $\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0$ (6.14)

Then (6.10) and (6.11) imply that **A** and **Φ** both satisfy the wave equations

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi = -\rho/\varepsilon_o \tag{6.15}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = -\mu_o\mathbf{J} \tag{6.16}$$

Question: With the Lorentz gauge condition, can you derive **A** and **Φ** uniquely from **E** and **B**?

Answer: No. It's possible to make a gauge transformation of the form (6.12) and (6.13) that preserves the Lorentz gauge condition. Substituting (6.12) and (6.13) in (6.14) gives

$$\nabla \cdot (\mathbf{A} + \nabla\Lambda) + \frac{1}{c^2}\frac{\partial}{\partial t}(\Phi - \frac{\partial\Lambda}{\partial t}) = 0$$

$$\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = 0 \tag{6.20}$$

Any solution to the homogeneous wave equation can be used to generate a gauge transformation that retains consistency with the Lorentz gauge condition

Choice 2. In Coulomb gauge $\nabla \cdot \mathbf{A} = 0$, (6.10) becomes

$$\nabla^2\Phi(\mathbf{x}, \mathbf{t}) = -\rho(\mathbf{x}, \mathbf{t})/\varepsilon_o$$

This is the Coulomb potential for the instantaneous charge density. Once Φ is calculated, one solves for **A**

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = \frac{1}{c^2}\frac{\partial\nabla\Phi}{\partial t} - \mu_o\mathbf{J}$$

That is all I will say about gauge conditions for the moment. There is more to say about the question of whether the potentials are measurable in quantum mechanics. That question involves the Aharonov-Bohm effect, but it is most convenient to postpone the discussion of that until we get to Chapter 12 of Jackson.

6.4 GREEN'S FUNCTIONS FOR THE WAVE EQUATION

The Coulomb and Biot-Savart laws are often very useful ways to find **E** and **B** in electrostatics and magnetostatics. They are essentially Green's functions.

For the time-dependent situation we are seeking solutions to the wave equation for **E** and **B**. The same is true for **A** and Φ in the Lorentz gauge. This leads us to consider the Green's function for the wave equation, which satisfies

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\boldsymbol{x}, \mathrm{t}; \mathbf{x}', t') = -4\pi\, \delta^{(3)}(\boldsymbol{x}-\mathbf{x}')\, \delta(t-t') \tag{6.41}$$

G represents the field of a source that exists only at points $\mathbf{x}'$, time t'. Jackson solves for *G* using Fourier transforms. I'm going to use a quicker and dirtier approach, which starts from my knowledge that the electrostatic Green's function, the solution to

$$\nabla^2 g(\mathbf{x},\mathbf{x}') = -4\pi\delta^{(3)}(\mathbf{x}-\mathbf{x}') \tag{X6.7}$$

is

$$g(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} \tag{X6.8}$$

We could multiply each side of (X6.7) by a delta function in time to obtain

$$\nabla^2 g'(\mathbf{x},t;\, \mathbf{x}',\mathbf{t}') = -4\pi\delta^{(3)}(\mathbf{x}-\mathbf{x}')\,\delta(t-t') \tag{X6.9}$$

$$g'(\mathbf{x},t;\, \mathbf{x}',t') = \frac{\delta(t-t')}{|\mathbf{x}-\mathbf{x}'|} \tag{X6.10}$$

We seek a *G* that is symmetric about the source—dependent on $\mathbf{x}$ and $\mathbf{x}'$ only through $R = |\mathbf{x}-\mathbf{x}'|$. It also depends on t and t′ only through $t-t' = \tau$

$$G(\mathbf{x},t;\, \mathbf{x}',t') = G(R,\tau) \tag{X6.11}$$

For $R > 0$, (6.41) becomes

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial\tau^2}\right) G(R,\tau) = 0 \tag{X6.12}$$

or

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial G}{\partial R}\right) - \frac{1}{c^2}\frac{\partial^2 G}{\partial\tau^2} = 0 \tag{X6.13}$$

Let

$$G(R,\tau) = \frac{f(R,\tau)}{R} \tag{X6.14}$$

Substituting (X6.14) in (X6.13) gives

$$\frac{\partial^2 f}{\partial R^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial\tau^2} = 0 \tag{X6.15}$$

which is the 1-D wave equation. The general solution to that equation is

$$f = f_1\left(\tau - \frac{R}{c}\right) + f_2\left(\tau + \frac{R}{c}\right) \tag{X6.16}$$

Therefore G must have the form

$$G = \frac{f_1\left(\tau - \frac{R}{c}\right) + f_2\left(\tau + \frac{R}{c}\right)}{R} \tag{X6.17}$$

On the other hand, comparison of (6.41), (X6.9) and (X6.10) implies that

$$\lim_{c\to\infty} G(R,\tau) = g'(R,\tau) = \frac{\delta(\tau)}{R} \tag{X6.18}$$

We can satisfy (X6.17) and (X6.18) by choosing

$$G = \frac{K\delta\left(\tau - \frac{R}{c}\right) + (1-K)\,\delta\left(\tau + \frac{R}{c}\right)}{R} \tag{X6.19}$$

where K = constant.

There are two independent solutions to (6.41). Any linear combination of them is a solution.

"Retarded Green's Function"

$$G_{\text{ret}}(\mathbf{x},t;\,\mathbf{x}',t') = \frac{\delta\left(t - t' - \frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right)}{|\boldsymbol{x}-\boldsymbol{x}'|} \tag{6.44}$$

"Advanced Green's Functions"

$$G_{\text{adv}}(x,t;\,\mathbf{x}',t') = \frac{\delta\left(t - t' + \frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right)}{|\boldsymbol{x}-\boldsymbol{x}'|}$$

Both of these are solutions to (6.41) for $R > 0$. If you don't trust my derivation, you can check the behavior at $R = 0$ by integrating (6.41) over a sphere of radius ε, and letting $\varepsilon \to 0$, to verify that the two proposed Green's functions work.

Physical Interpretations of the 2 Green's Functions:

1. Retarded Green's Function

A source at $\mathbf{x}', t'$ causes a signal at $\mathbf{x}$ at time $t' + \frac{|\mathbf{x}-\mathbf{x}'|}{c}$. That's very intuitive. The signal propagates at c from the source to the observer.

2. Advanced Green's Function

Source at $\mathbf{x}'$, t' causes a signal that is observed at $\mathbf{x}$ at time $t = t' - \frac{|\mathbf{x}-\mathbf{x}'|}{c}$. You see the signal from the source before the source exists. That's somewhat less intuitive. The advanced G represents a signal that propagates from very large distances and converges on the source just when it turns on.

These 2 solutions to Maxwell's equations exist because the equations are invariant under time reversal

$$t \to -t$$
$$\mathbf{J} \to -\mathbf{J}$$
$$\rho \to \rho$$
$$\mathbf{E} \to \mathbf{E}$$
$$\mathbf{B} \to -\mathbf{B}.$$

Mathematically, the incoming wave solution is just as valid as the outgoing wave solution.

The standard procedure is to reject the advanced solutions as unphysical and set

$$G = G_{\text{ret}}$$

and that's what we'll do in this course. However, theorists have long been uncomfortable with this and there have been efforts to construct theories of electrodynamics that treat the solutions differently.

There is a related difficulty with the classical theory of electrodynamics, namely the radiation-reaction problem for a point charge. Suppose an electron accelerates under the influence of some known force—a magnetic field, for example. An accelerating charge radiates, losing energy. So an electron circling a magnetic field will gradually lose energy. There must be a radiation-reaction force on the electron. But what exerts the radiation-reaction force on the electron? There's nothing else around. How can the electron exert a force on itself? (At the end of the course, we'll discuss approaches to calculating radiation-reaction force (Ch. 16.) However, none of the proposed solutions is fully satisfactory, as far as I know.)

One famous effort at dealing with the time-reversal problem and the radiation-reaction problem is the Wheeler-Feynman approach [*Rev. Mod. Phys., 17,* 157, 1945; *21*, 425, 1949]. (This was essentially Feynmann's Ph.D. thesis.) Wheeler and Feynmann set

$$\mathrm{G} = \frac{1}{2}\ \mathrm{G}_{\mathrm{ret}} + \frac{1}{2}\ \mathrm{G}_{\mathrm{adv}}$$

Then they put lots of free electrons at large distances, where they act to partly absorb and partly reflect the original radiation. Model calculations give, for the field at the source due to the distant electrons

$$\frac{1}{2}(\mathrm{ret} - \mathrm{adv})$$

When they add 1/2 ($\mathrm{G}_{\mathrm{ret}} + \mathrm{G}_{\mathrm{adv}}$) directly from the source, they find, for the observed field

$$\frac{1}{2}(\mathrm{ret} + \mathrm{adv}) + \frac{1}{2}(\mathrm{ret} - \mathrm{adv}) = \mathrm{ret}$$

So their electrodynamics generally gives the same answer as conventional Jackson-type electrodynamics.

How this works

- Electron accelerates at t'.
- Electrons at r feel retarded-wave effects at $t = t' + r/c$ + that jiggling causes re-radiation
- Advanced re-radiation arrives at the original electron at t'.

- Retarded re-radiation arrives at $t' + 2r/c$—that is so far in the future that it doesn't matter.
- Also consider advanced radiation from original electron + retarded solution from the distant electrons.
- The theory assumed that there were enough electrons at great distances to completely absorb the outgoing radiation.

Advantage of this: The radiation-reaction force on the original electron comes out right—and in a very natural way—it's the force exerted on the original electron by the shell of distant electrons.

The reason that you aren't learning Wheeler-Feynman electrodynamics is that it is inconsistent with modern cosmology:

Wheeler-Feynman had to make their distant electron shell stationary.

If it were moving out with the expanding universe then the signal received back would be Doppler shifted down—wouldn't cancel.

Hoyle and collaborators worked Wheeler-Feynman electrodynamics for the steady-state universe and claimed it worked. However, the steady-state cosmology fell from favor with the discovery of the cosmic microwave background.