PHYS 532 L4

Recall from last time, a localized charge distribution $\rho(\underline{x}')$ produces a dipole moment

$$\underline{p} = \int \rho(\underline{x}')\, \underline{x}'\, d^3x' = \text{vector}$$

If a medium has number density of species s, N_s, each with dipole moment $\langle \underline{p}_s \rangle$ averaged over a macroscopic volume, we can define the polarization vector

$$\underline{P} = \sum_s N_s \langle \underline{p}_s \rangle$$

$$= \text{dipole moment density.}$$

This polarization produces a potential

$$\Phi(\underline{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \, \frac{\underline{P}(\underline{x}') \cdot (\underline{x}-\underline{x}')}{|\underline{x}-\underline{x}'|^3}$$

$$= \frac{1}{4\pi\epsilon_0} \int d^3x' \, \underline{P}(\underline{x}') \cdot \nabla' \frac{1}{|\underline{x}-\underline{x}'|} \qquad (X4.4)$$

Now,

$$\nabla' \cdot \left(\frac{\underline{P}(\underline{x}')}{|\underline{x}-\underline{x}'|} \right) = \frac{\nabla' \cdot \underline{P}(\underline{x}')}{|\underline{x}-\underline{x}'|} + \underline{P}(\underline{x}') \cdot \nabla' \frac{1}{|\underline{x}-\underline{x}'|}$$

So (X4.4) can be written

$$\Phi(\underline{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \left[\nabla' \cdot \frac{\underline{P}(\underline{x}')}{|\underline{x}-\underline{x}'|} - \frac{\nabla' \cdot \underline{P}(\underline{x}')}{|\underline{x}-\underline{x}'|} \right] \qquad (X4.5)$$

If the volume of integration encloses the dielectric completely, the first term gives zero, by the divergence theorem, because $\underline{P} = 0$ outside the dielectric.

If we include the effect of any free charge density $\rho(\underline{x}')$ on the potential, (X 4.5) gives

$$\Phi(\underline{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \, \frac{\rho(\underline{x}') - \nabla' \cdot \underline{P}(\underline{x}')}{|\underline{x} - \underline{x}'|} \qquad (4.32)$$

There is a polarization charge density

$$\rho\,(\text{polarization}) = -\nabla \cdot \underline{P}$$

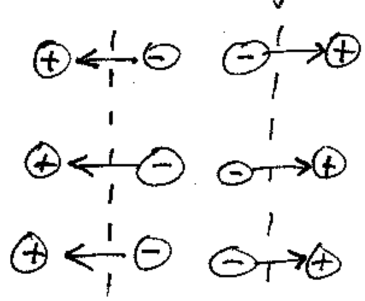
(4.32) leads to

$$\boxed{\nabla \cdot \underline{E} = \frac{1}{\epsilon_0}\left(\rho - \nabla \cdot \underline{P}\right)} \qquad (4.33)$$

free charge ↗ ↖ polarization charge

by the same argument that leads from (1.11) to (1.13).

Physical meaning of $-\nabla \cdot \underline{P}$:



$\nabla \cdot \underline{P} > 0$

central region has excess negative charge.

Re-arranging (4.33) gives

$$\nabla \cdot (\epsilon_0 \underline{E} + \underline{P}) = \rho \qquad (\times 4.7)$$

It is conventional to define an <u>electric</u> <u>displacement</u> <u>vector</u> $\underline{D} = \epsilon_0 \underline{E} + \underline{P}$ (4.34) such that

$$\nabla \cdot \underline{D} = \rho \quad \text{(free charge density)} \qquad (4.35)$$

If the medium is linear and isotropic,

$$\underline{P} = \epsilon_0 \chi_e \underline{E} \qquad (4.36)$$

which defines the <u>electric</u> <u>susceptibility</u> χ_e.

If the medium is linear but not isotropic, then χ_e is a tensor ($P_i = \epsilon_0 \chi_{ij} E_j$), but we will assume an isotropic medium and

$$\underline{D} = \epsilon_0 \underline{E} + \epsilon_0 \chi_e \underline{E} = \epsilon_0 (1 + \chi_e) \underline{E} \qquad (\times 4.8)$$

Define the <u>electric</u> <u>permittivity</u> $\epsilon = \epsilon_0 (1 + \chi_e)$ (4.38)

Then the <u>dielectric</u> <u>constant</u> is defined by

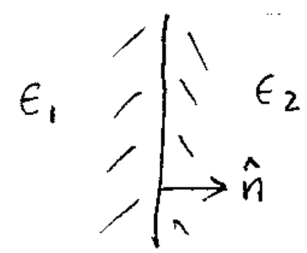
$$\frac{\epsilon}{\epsilon_0} = 1 + \chi_e \quad (> 1 \text{ for dielectric}).$$

and $\underline{D} = \epsilon \underline{E}$ (<u>constitutive</u> <u>relation</u>) (4.37)

$$\nabla \cdot \underline{E} = \rho / \epsilon \qquad (4.39)$$

Recall that the multipole expansion is valid only outside the charge distribution that produces the multipoles. Using just the dipole term does not accurately represent the effects of nearby atoms/molecules. We'll consider the near field next time.

<u>Boundary conditions at an interface</u> between two dielectrics:



$\nabla \times \underline{E} = 0$ in electrostatics $\Rightarrow \hat{n} \times (\underline{E}_1 - \underline{E}_2) = 0$

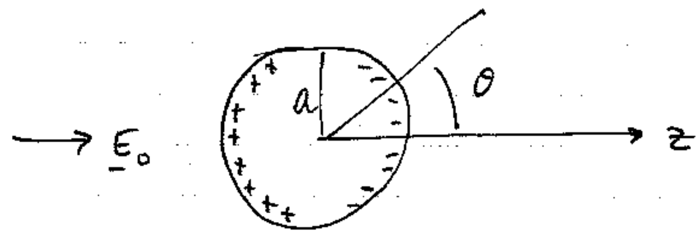
(tangential component of $\underline{E}$ is conserved.)

No free charge on interface $\Rightarrow \hat{n} \cdot (\underline{D}_1 - \underline{D}_2) = 0$

(normal component of $\underline{D}$ is conserved.)

A boundary-value problem with dielectrics:

Consider a vacuum spherical hole inside an otherwise uniform dielectric with permitivity ϵ, subject to an external electric field $\underline{E}_0$ in the $\hat{z}$ direction. How is $\underline{E}$ perturbed by the hole?



The boundary condition $\nabla \cdot \underline{D} = 0$ gives

$$\epsilon E_r(r = a+\delta) - \epsilon_0 E_r(r = a-\delta) = 0 \qquad (X4.9)$$

$$\text{or} \quad \epsilon \frac{\partial \Phi}{\partial r}(a+\delta) = \epsilon_0 \frac{\partial \Phi}{\partial r}(a-\delta) \qquad (X4.9)$$

Inside and outside the sphere, $\nabla^2 \Phi = 0$.
General solution in spherical coordinates is

$$\Phi(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(a_{lm} r^{l} + b_{lm} r^{-l-1}\right) Y_{lm}(\theta, \varphi) \qquad (3.61)$$

where Y_{lm} = spherical harmonics.

Axial symmetry (no φ-dependence) $\Rightarrow$ only $m=0$ terms are non-zero, and

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} \left(c_l r^l + d_l r^{-l-1}\right) P_l(\cos\theta)$$

where $P_l(\cos\theta)$ = Legendre polynomial.

For $r<a$, Φ must be regular at origin
$\Rightarrow d_l = 0$, $r<a$.

For $r>a$, Φ must be regular at infinity.
$\Rightarrow c_l = 0$, $r>a$.

Therefore, write

$$\Phi(r,\theta) = -E_0 r\cos\theta + \sum_\ell \begin{Bmatrix} c_\ell \left(\frac{r}{a}\right)^\ell \\ d_\ell \left(\frac{a}{r}\right)^{\ell+1} \end{Bmatrix} P_\ell(\cos\theta)$$

$$\text{for } \begin{Bmatrix} r < a \\ r > a \end{Bmatrix} \qquad (x4.10)$$

1ST term = uniform field at ∞.
Sum = field due to charges on sphere.

Continuity of Φ at $r=a \Rightarrow d_\ell = c_\ell$

Boundary condition (x4.9) gives

$$\epsilon\left[-E_0\cos\theta - \sum_\ell c_\ell \frac{\ell+1}{a} P_\ell(\cos\theta)\right] = \epsilon_0\left[-E_0\cos\theta + \sum_\ell c_\ell \frac{\ell}{a} P_\ell(\cos\theta)\right]$$

Orthogonality of Legendre polynomials $\Rightarrow$ $c_\ell = 0$ for $\ell \neq 1$. For $\ell = 1$, $P_1(\cos\theta) = \cos\theta$ and

$$\epsilon\left[-E_0\cos\theta - \frac{2c_1}{a}\cos\theta\right] = \epsilon_0\left[-E_0\cos\theta + \frac{c_1}{a}\cos\theta\right]$$

$$c_1\left(-\frac{2\epsilon}{a} - \frac{\epsilon_0}{a}\right) = (\epsilon - \epsilon_0)E_0$$

$$c_1 = -\frac{(\epsilon-\epsilon_0)E_0 a}{\epsilon_0 + 2\epsilon}$$

$$\Phi(r,\theta) = -E_0 r\cos\theta\left[1 + \left(\frac{\epsilon-\epsilon_0}{\epsilon_0+2\epsilon}\right)\begin{Bmatrix} 1, & r<a \\ \left(\frac{a}{r}\right)^3, & r>a \end{Bmatrix}\right]$$

(x 4.11)

Inside sphere, $\underline{E}_{in} = \underline{E}_0 \left(\frac{3\epsilon}{\epsilon_0 + 2\epsilon} \right)$ (4.59)

$E_{in} > E_0$ (if $\epsilon > \epsilon_0$)

Outside sphere, $\underline{E} = \underline{E}_0$ + dipole perturbation.