Recall from last time, a localized charge distribution \( p(x') \) produces a dipole moment

\[
\mathbf{p} = \int p(x') \, x' \, d^3x' = \text{vector}
\]

If a medium has number density of species \( s \), \( N_s \), each with dipole moment \( \langle p_s \rangle \) averaged over a macroscopic volume, we can define the polarization vector

\[
\mathbf{P} = \frac{\varepsilon}{\varepsilon_0} N_s \langle p_s \rangle
\]

= dipole moment density.

This polarization produces a potential

\[
\Phi(x) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \, \frac{p(x') \cdot (x - x')}{|x - x'|^3}
\]

\[
= \frac{1}{4\pi\varepsilon_0} \int d^3x' \, p(x') \cdot \nabla' \frac{1}{|x - x'|} \quad \text{(X4.4)}
\]

Now,

\[
\nabla' \left( \frac{p(x')}{|x - x'|} \right) = \nabla' \cdot \frac{p(x')}{|x - x'|} + \frac{p(x') \cdot \nabla' \frac{1}{|x - x'|}}{|x - x'|}
\]

So \( \text{(X4.4)} \) can be written

\[
\Phi(x) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \left[ \nabla' \cdot \frac{p(x')}{|x - x'|} - \frac{\nabla' \cdot p(x')}{|x - x'|} \right] \quad \text{(X4.5)}
\]
If the volume of integration encloses the dielectric completely, the first term gives zero, by the divergence theorem, because \( \rho = 0 \) outside the dielectric.

If we include the effect of any free charge density \( \rho(x') \) on the potential, (4.32) gives

\[
\Phi(x) = \frac{1}{4\pi\varepsilon_0} \int d^3 x' \frac{\rho(x') - \nabla' \cdot P(x')}{|x - x'|} \tag{4.32}
\]

There is a polarization charge density

\[
\rho_{\text{polarization}} = -\nabla \cdot P
\]

(4.32) leads to

\[
\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} (\rho - \nabla \cdot \mathbf{P}) \tag{4.33}
\]

Free charge \( \rho \) polarization charge

by the same argument that leads from (1.11) to (1.13).

Physical meaning of \( -\nabla \cdot P \):

\[
\begin{align*}
\text{if} & \quad \nabla \cdot P > 0 \\
\text{central region has} & \quad \text{excess negative charge.}
\end{align*}
\]
Re-arranging (4.33) gives
\[ \nabla \cdot (\varepsilon_0 E + P) = \rho \]  \hspace{1cm} (x.4.7)

It is conventional to define an \underline{electric displacement vector} \( \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \) \hspace{1cm} (4.34)
such that
\[ \nabla \cdot \mathbf{D} = \rho \hspace{1cm} \text{(free charge density)} \hspace{1cm} (4.35) \]

If the medium is linear and \underline{isotropic},
\[ \mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} \] \hspace{1cm} (4.36)

which defines the \underline{electric susceptibility} \( \chi_e \).

If the medium is linear but not \underline{isotropic},
then \( \chi_e \) is a tensor \( \mathbf{P}_i = \varepsilon_0 \chi_{ij} \mathbf{E}_j \), but
we will assume an \underline{isotropic medium} and
\[ \mathbf{D} = \varepsilon_0 \mathbf{E} + \varepsilon_0 \chi_e \mathbf{E} = \varepsilon_0 (1 + \chi_e) \mathbf{E} \] \hspace{1cm} (x.4.8)

Define the \underline{electric permittivity} \( \varepsilon = \varepsilon_0 (1 + \chi_e) \) \hspace{1cm} (4.38)

Then the \underline{dielectric constant} is defined by
\[ \frac{\varepsilon}{\varepsilon_0} = 1 + \chi_e \hspace{1cm} (> 1 \text{ for dielectric}) \]

and \[ \mathbf{D} = \varepsilon \mathbf{E} \hspace{1cm} \text{(constitutive relation)} \hspace{1cm} (4.37) \]
\[ \nabla \cdot \mathbf{E} = \rho / \varepsilon \] \hspace{1cm} (4.39)
Recall that the multipole expansion is valid only outside the charge distribution that produces the multipoles. Using just the dipole term does not accurately represent the effects of nearby atoms/molecules. We'll consider the near field next time.

**Boundary conditions at an interface between two dielectrics:**

\[
\begin{array}{c}
E_1 \\
\downarrow \\
\downarrow \\
\downarrow \\
\hat{n} \\
\end{array}
\begin{array}{c}
\varepsilon_2 \\
\end{array}
\]

\[\nabla \times E = 0 \text{ in electrostatics } \Rightarrow \hat{n} \times (\varepsilon_1 - \varepsilon_2) = 0 \]

(tangential component of \( E \) is conserved.)

No free charge on interface \[\Rightarrow \hat{n} \cdot (\nabla_1 - \nabla_2) = 0 \]

(normal component of \( \nabla \) is conserved.)

A boundary-value problem with dielectrics:

Consider a vacuum spherical hole inside an otherwise uniform dielectric with permittivity \( \varepsilon_1 \), subject to an external electric field \( E_0 \) in the \( \hat{z} \) direction. How is \( E \) perturbed by the hole?
The boundary condition \( \nabla \cdot \mathbf{E} = 0 \) gives

\[
\varepsilon E_r(r = a + \delta) - \varepsilon_0 E_r(r = a - \delta) = 0 \quad (4.9)
\]

or

\[
\varepsilon \frac{\partial E_r}{\partial r}(a + \delta) = \varepsilon_0 \frac{\partial E_r}{\partial r}(a - \delta) \quad (4.9)
\]

Inside and outside the sphere, \( \nabla^2 \Phi = 0 \). General solution in spherical coordinates is

\[
\Phi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left( a_{lm} r^l + b_{lm} r^{-l-1} \right) Y_{lm}(\theta, \phi)
\]

where \( Y_{lm} \) = spherical harmonics.

Axial symmetry (no \( \phi \)-dependence) \( \Rightarrow \) only \( m = 0 \) terms are non-zero, and

\[
\Phi(r, \theta) = \sum_{l=0}^{\infty} \left( a_{l} r^l + b_{l} r^{-l-1} \right) P_l(\cos \theta)
\]

where \( P_l(\cos \theta) \) = Legendre polynomial.

For \( r < a \), \( \Phi \) must be regular at origin

\( \Rightarrow \) \( a_l = 0 \), \( r < a \).

For \( r > a \), \( \Phi \) must be regular at infinity

\( \Rightarrow \) \( b_l = 0 \), \( r > a \).
Therefore, write

\[ \Phi(r, \theta) = -E_0 r \cos \theta + \sum \left\{ \frac{C_l (\frac{r}{a})^l}{d_l (\frac{a}{r})^{l+1}} \right\} P_l(\cos \theta) \]

\[ \text{for } \begin{cases} r < a \\{ r > a \} \end{cases} \]

(x4.10)

1st term = uniform field at \( \infty \).
Sum = field due to charges on sphere.
Continuity of \( \Phi \) at \( r = a \) \( \Rightarrow d_l = C_l \)

Boundary condition (x4.9) gives

\[ \epsilon \left[ -E_0 \cos \theta - \sum \frac{C_l}{d_l} \frac{l+1}{a} P_l(\cos \theta) \right] = \epsilon_0 \left[ -E_0 \cos \theta + \sum \frac{C_l}{d_l} \frac{1}{a} P_l(\cos \theta) \right] \]

Orthogonality of Legendre polynomials \( \Rightarrow C_l = 0 \) for \( l \neq 1 \). For \( l=1 \), \( P_1(\cos \theta) = \cos \theta \) and

\[ \epsilon \left[ -E_0 \cos \theta - \frac{2C_1}{a} \cos \theta \right] = \epsilon_0 \left[ -E_0 \cos \theta + \frac{C_1}{a} \cos \theta \right] \]

\[ C_1 \left( -\frac{2\epsilon}{a} - \frac{\epsilon_0}{a} \right) = (\epsilon - \epsilon_0) E_0 \]

\[ C_1 = -\frac{(\epsilon - \epsilon_0) E_0 a}{\epsilon_0 + 2\epsilon} \]

\[ \Phi(r, \theta) = -E_0 r \cos \theta \left[ 1 + \left( \frac{\epsilon - \epsilon_0}{\epsilon_0 + 2\epsilon} \right) \left\{ \frac{1}{r^3}, r > a \right\} \right] \]

(x4.11)
Inside sphere,  \[ E_{in} = E_0 \left( \frac{3\epsilon}{\epsilon_0 + 2\epsilon} \right) \]  (4.59)

\[ E_{in} > E_0 \text{ (if } \epsilon > \epsilon_0) \]

Outside sphere,  \[ E = E_0 + \text{dipole perturbation.} \]