Radiation from accelerating charge (from L36):

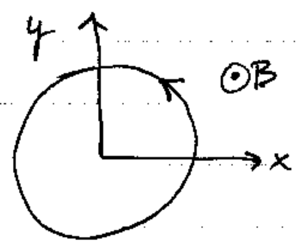
$$\underline{E}(\underline{x},t) = \frac{q}{cR}\left\{ \frac{\hat{n}\times[(\hat{n}-\underline{\beta})\times\dot{\underline{\beta}}\,]}{|1-\hat{n}\cdot\underline{\beta}|^3} \right\}_{ret} \qquad (14.14)$$

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c}\,\frac{|\hat{n}\times(\hat{n}-\underline{\beta})\times\dot{\underline{\beta}}|^2}{(1-\hat{n}\cdot\underline{\beta})^5} \qquad (14.38)$$

Example 1: Cyclotron radiation.

(Non-relativistic particle gyrating about $\underline{B} = B_0\hat{z}$)

NR $\Rightarrow$ $\beta \to 0$ in (14.14) and (14.38)



$$x = a\cos\Omega t' \qquad \ddot{x} = -\Omega^2 x$$

$$y = a\sin\Omega t' \qquad \ddot{y} = -\Omega^2 y$$

$$\Omega = \frac{qB}{\gamma mc} \to \frac{qB}{mc}$$

$$\dot{\underline{\beta}} = -\frac{\Omega^2 a}{c}\left(\hat{x}\cos\Omega t' + \hat{y}\sin\Omega t'\right)$$

Let $\hat{n} = \hat{z}\cos\theta + \hat{x}\sin\theta$ (θ = angle between $\underline{B}$ and line of sight)

$$\to \hat{n}\times(\hat{n}\times\dot{\underline{\beta}}) = -(\hat{z}\cos\theta+\hat{x}\sin\theta)\times\left[(\hat{z}\cos\theta+\hat{x}\sin\theta)\times\frac{\Omega^2 a}{c}(\hat{x}\cos\Omega t'+\hat{y}\sin\Omega t')\right]$$

$$= -\frac{\Omega^2 a}{c}(\hat{z}\cos\theta+\hat{x}\sin\theta)\times\left[-\hat{x}\cos\theta\sin\Omega t' + \hat{y}\cos\theta\cos\Omega t' + \hat{z}\sin\theta\sin\Omega t'\right]$$

$$= -\frac{\Omega^2 a}{c}\Big[\cos\theta(-\hat{y}\cos\theta\sin\Omega t' - \hat{x}\cos\theta\cos\Omega t')$$

$$+\sin\theta(\hat{z}\cos\theta\cos\Omega t' - \hat{y}\sin\theta\sin\Omega t')\Big]$$

$$\hat{n}\times(\hat{n}\times\dot{\underline{\beta}}) = \frac{\Omega^2 a}{c}\left[\hat{x}\cos^2\theta\cos\Omega t' + \hat{y}\sin\Omega t' - \hat{z}\cos\theta\sin\theta\cos\Omega t'\right]$$

At $\theta = 0$ (looking along $-\underline{B}$), one sees

$$\underline{E} \propto \hat{x}\cos\Omega t' + \hat{y}\sin\Omega t' \quad = \text{circular polarization}$$
$$= 2 \text{ dipoles, } 90^\circ \text{ out of phase}$$

At $\theta = \pi/2$ (looking $\perp$ $\underline{B}$) one sees

$$\underline{E} \propto \hat{y}\sin\Omega t' = \text{linear polarization} = 1 \text{ dipole}$$

From (14.38), the angular distribution is

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c}\left|\hat{n}\times(\hat{n}\times\dot{\underline{\beta}})\right|^2$$

$$= \frac{q^2\Omega^4 a^2}{4\pi c^3}\left[\cos^4\theta\cos^2\Omega t' + \sin^2\Omega t' + \cos^2\theta\sin^2\theta\cos^2\Omega t'\right]$$

$$= \frac{q^2\Omega^4 a^2}{4\pi c^3}\left[\cos^2\theta\cos^2\Omega t' + \sin^2\Omega t'\right]$$

$$= \frac{q^2\Omega^4 a^2}{4\pi c^3}\left[1 - \sin^2\theta\cos^2\Omega t'\right]$$

Averaging over $\Omega t'$ gives

$$\boxed{\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{q^2\Omega^4 a^2}{4\pi c^3}\left[1 - \tfrac{1}{2}\sin^2\theta\right]} \qquad (x14.13)$$

Integrating over solid angle $d\Omega = 2\pi\sin\theta\, d\theta$ gives

$$P = \frac{2q^2\Omega^4 a^2}{3c^3} = \frac{2q^2|\dot{\underline{v}}|^2}{3c^3} \qquad (14.22)$$

the Larmor formula (again).

Example 2: Extreme relativistic limit ($\gamma >> 1$)

$$\gamma^2 = \frac{1}{1-\beta^2} \Rightarrow \beta^2 = 1 - \frac{1}{\gamma^2}$$

$$\Rightarrow \beta = \sqrt{1-\frac{1}{\gamma^2}} \approx 1 - \frac{1}{2\gamma^2} \qquad (\gamma^2 >> 1)$$

If θ = angle between $\hat{n}$ and $\underline{\beta}$, then

$$1 - \hat{n}\cdot\underline{\beta} = 1 - \beta\cos\theta \approx 1 - \left(1 - \frac{1}{2\gamma^2}\right)\cos\theta$$

$$\therefore\ 1 - \hat{n}\cdot\underline{\beta} \sim 1 \quad \text{for} \quad \cos\theta \lesssim 1$$

but $1 - \hat{n}\cdot\underline{\beta} \approx \frac{1}{2\gamma^2} << 1$ for $\cos\theta = 1$ $\quad(\theta = 0)$

For small θ, $\cos\theta \approx 1 - \frac{\theta^2}{2}$ and

$$1 - \hat{n}\cdot\underline{\beta} \approx 1 - \left(1 - \frac{1}{2\gamma^2}\right)\left(1 - \frac{\theta^2}{2}\right) \approx \frac{\theta^2}{2} + \frac{1}{2\gamma^2} << 1$$

Since the intensity distribution (14.38) is proportional to a large negative power of $(1 - \hat{n}\cdot\beta)$, it is generally peaked at a small value of $\theta_0 \sim 1/\gamma$ with a peak width $\Delta\theta \sim 1/\gamma$.

The exact angular distribution is sensitive to the angle between $\underline{\beta}$ and $\underline{\dot{\beta}}$. There are 2 limiting cases:

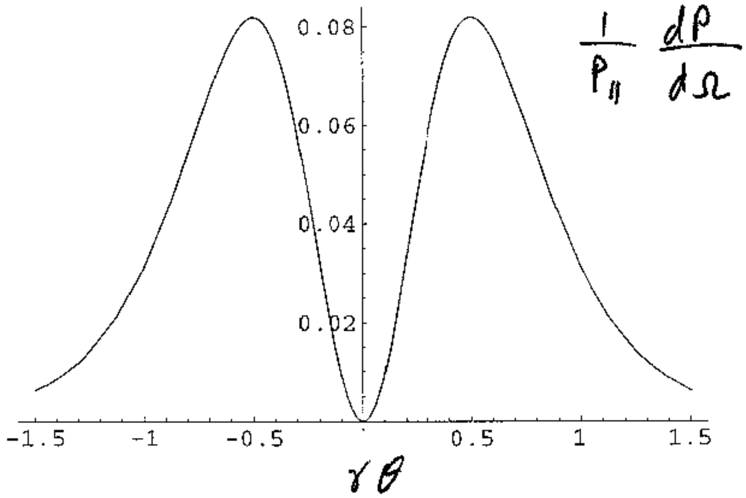
A. Parallel acceleration, $\underline{\dot{\beta}} \parallel \underline{\beta}$

B. Perpendicular acceleration, $\underline{\dot{\beta}} \perp \underline{\beta}$

For parallel acceleration, Jackson gets

$$\frac{dP}{d\Omega} = \frac{8}{\pi}\frac{q^2 \dot{v}^2}{c^3}\gamma^8 \frac{(\gamma\theta)^2}{(1+\gamma^2\theta^2)^5} \qquad (14.41)$$

which is plotted below as a function of $\gamma\theta$:



$$P_\parallel = \frac{8}{\pi}\frac{q^2\dot{v}^2}{c^3}\gamma^8$$

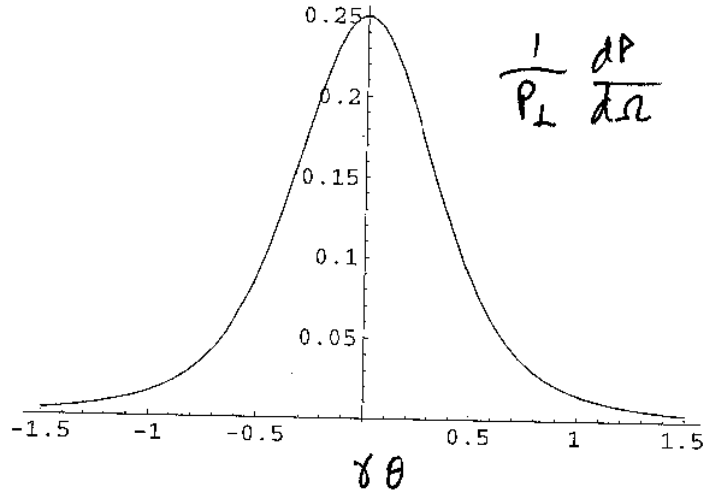
(equivalent to Jackson Fig. 14.5)

Thus, parallel acceleration gives a hollow emission cone peaked at $\theta_{max} = \frac{1}{2\gamma}$ with width $\sim \frac{1}{\gamma}$.

For perpendicular acceleration, Jackson gets

$$\frac{dP}{d\Omega} = \frac{2}{\pi}\frac{q^2\dot{v}^2}{c^3}\gamma^6\left[\frac{1 - \frac{4\gamma^2\theta^2\cos^2\phi}{(1+\gamma^2\theta^2)^2}}{(1+\gamma^2\theta^2)^3}\right] \qquad (14.45)$$

Which is plotted below for the average value $\langle\cos^2\phi\rangle = \frac{1}{2}$. (The angle ϕ is defined in Jackson's Fig. 14.6. The peak is somewhat broader for $\cos^2\phi = 0$ and somewhat narrower for $\cos^2\phi = 1$.)



$P_\perp = P_\parallel / \gamma^2$

So, for perpendicular acceleration, the radiation is confined to a single beam centered at $\theta = 0$ (the direction of motion), with (again) a width $\Delta\theta \sim 1/\gamma$.

This strong forward beaming is a relativistic effect, caused by the Lorentz transformations of $\underline{E}$ and $\underline{B}$. (From a Galilean transformation, the largest beaming angle is $\sim 45°$.)

The comparison of the two figures above is valid for the angular distribution of radiation, but not for its magnitude. Both are normalized by $|\dot{v}|^2$, which is very different for $\parallel$ vs. $\perp$ acceleration.

For $\parallel$ acceleration, caused by an electric field $\underline{E}$, we have

$$q\underline{E} = \frac{d\underline{p}}{dt} = \frac{d}{dt}(\gamma m \underline{v}) = m\frac{d}{dt}\left(\frac{v}{\sqrt{1 - v^2/c^2}}\right)$$

$$q\underline{E} = m\,\frac{\underline{\dot{v}}(1-v^2/c^2)^{1/2} - \underline{v}\left(\tfrac{1}{2}(1-v^2/c^2)^{-1/2}(-2\,\tfrac{v\dot{v}}{c^2})\right)}{1-v^2/c^2}$$

$$= m\underline{\dot{v}}\,\gamma^3\left[(1-v^2/c^2) + \frac{v^2}{c^2}\right] = m\gamma^3\underline{\dot{v}}$$

so $$|\underline{\dot{v}}|^2_{\parallel} = \frac{1}{m^2\gamma^6}\,q^2E^2 \qquad (X14.14)$$

while for perpendicular acceleration, caused by a magnetic field $\underline{B}$, we have

$$\dot{v} = \Omega v \approx \Omega c = \frac{qBc}{\gamma mc} = \frac{qB}{m\gamma} \quad \text{and}$$

$$|\underline{\dot{v}}|^2_{\perp} = \frac{q^2B^2}{m^2\gamma^2}. \qquad (X14.15)$$

For parallel acceleration (first figure above), the normalizing value is

$$P_{\parallel} = \frac{8}{\pi}\frac{q^2\dot{v}^2}{c^3}\gamma^8 = \frac{8}{\pi}\frac{q^4E^2}{m^2c^3}\gamma^2 \qquad (X14.16)$$

While for $\perp$ acceleration (second figure) the normalizing value is

$$P_{\perp} = \frac{8}{\pi}\frac{q^2\dot{v}^2}{c^3}\gamma^6 = \frac{8}{\pi}\frac{q^4B^2}{m^2c^3}\gamma^4 \qquad (X14.17)$$

so for "equal forces" $(E=B)$,

$$\frac{P_{\perp}}{P_{\parallel}} = \gamma^2$$

and $\perp$ acceleration is γ^2 more effective than $\parallel$ acceleration in producing radiation.

Example 3. Synchrotron radiation.

The ultrarelativistic counterpart to cyclotron radiation. Because of the efficiency of $\perp$ acceleration versus $\parallel$ acceleration (factor of γ^2), synch. is one of nature's (and man's) most powerful radiation mechanisms. The total power, integrating $\frac{dP}{d\Omega}$ (14.45) over $d\Omega$, is

$$P = \frac{2}{3}\frac{q^2\gamma^4}{c^3}|\dot{v}|^2 \qquad (14.46)$$

or, combining with (x14.15),

$$\boxed{P = \frac{2}{3}\frac{q^4 B^2}{m^2 c^3}\gamma^2} \qquad (x14.18)$$

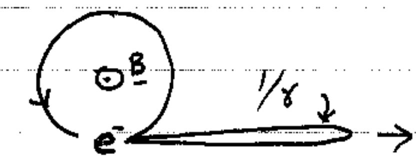
This is the power emitted by a single electron. Usually, of course, one observes the emission from many electrons at once.

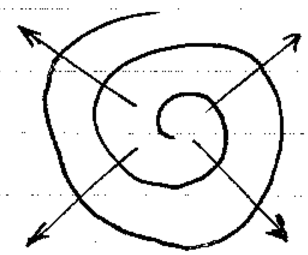
Cyclotron radiation (nonrelativistic) is emitted at the cyclotron frequency $\omega_B = \frac{qB}{\gamma mc}$. Synchrotron radiation, by contrast, is emitted in a broad band of frequencies extending up to

$$\omega \sim \omega_c = \frac{3}{2}\omega_B\gamma^3 = \frac{3qB}{2mc}\gamma^2 \qquad (14.81)$$

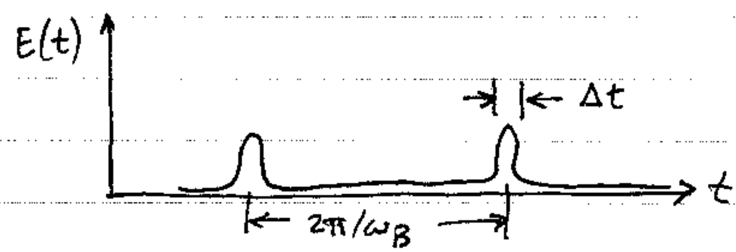
Calculating the frequency spectrum is complicated (Jackson §14.6), but the origin of the cut-off frequency ω_c (14.81) can be understood qualitatively as follows.

A given electron emits a thin beam ($\Delta\theta \sim 1/\gamma$) in its instantaneous direction of motion. This gives rise to a spiral-shaped pulse of radiation:





A fixed observer looking $\perp \underline{B}$ sees a short pulse of radiation once per gyration period $2\pi/\omega_B$:



In the electron's frame, the pulse width is

$$\Delta t' \sim \frac{2\pi}{\omega_B}\frac{2\Delta\theta}{2\pi} = \frac{2\Delta\theta}{\omega_B} \sim \frac{2}{\gamma\omega_B}$$

In the observer's frame this is

$$\Delta t = \frac{\partial t}{\partial t'}\Delta t' = (1-\hat{n}\cdot\underline{\beta})\frac{2}{\gamma\omega_B} \sim \frac{1}{2\gamma^2}\frac{2}{\gamma\omega_B} = \frac{1}{\gamma^3\omega_B}$$

so the highest frequency excited is $\omega_c \sim \gamma^3\omega_B$.

The spectrum (for a single electron) is actually a series of harmonics of ω_B extending up to $\omega_c \sim \gamma^3\omega_B$.