Radiation from an accelerating point charge.

Start with the Jefimenko (retarded-time) versions of the Coulomb and Biot-Savart integrals, as derived in Jackson (§6.5) and in L9:

\[ E(x, t) = \int d^3x' \left\{ \frac{\hat{R}}{R^2} \left[ \rho(x', t') \right]_{\text{ret}} + \frac{R}{cR} \left[ \frac{\partial \mathbf{P}(x', t')}{\partial t'} \right]_{\text{ret}} - \frac{i}{cR} \left[ \frac{\partial \mathbf{S}(x', t')}{\partial t'} \right]_{\text{ret}} \right\} \]

where subscript "ret" means evaluated at \( t' = t - \frac{R}{c} \), and \( R = |x - x'|, \quad R = |\mathbf{R}| \).

The first term (\( \propto \frac{1}{R^2} \)) is not a radiation field, and at sufficiently large \( R \) it is negligible compared to the other two terms (\( \propto \frac{1}{R} \)). Throw it away.

Leaving off the subscript "ret", we note that

\[ \left[ \frac{\partial \mathbf{P}(x', t')}{\partial t'} \right]_{x'} = \frac{\partial \mathbf{P}}{\partial t} \left( \frac{\partial t'}{\partial t} \right)_{x'} = \frac{\partial \mathbf{P}}{\partial t} \frac{\partial t'}{\partial t} \left( t' - \frac{R}{c} \right) = \frac{\partial \mathbf{P}}{\partial t'} \quad (x14.1) \]

so (6.55) can be written

\[ E(x, t) = \frac{\partial}{\partial t} \int d^3x' \left[ \frac{\rho(x', t') \hat{R}}{cR} - \frac{\mathbf{S}(x', t')}{c^2 R} \right] \quad (x14.2) \]

This is good for any \( \rho(x', t') \), \( \mathbf{S}(x', t') \). In particular, for a point charge \( q \) on a specified trajectory \( x' = \gamma_q(t) \), we can substitute...
\[ p(x', t') = \rho \delta^{(3)}(x' - r_0(t')) \quad (x14.3) \]
\[ J(x', t') = p(x', t') \nu_0(t') \quad (x14.4) \]

where \( \nu_0(t') = \frac{dr_0}{dt'} \).

The delta functions in \((x14.3, x14.4)\) make the integrals in \((x14.2)\) easy, right? Well, yes, but we need to be careful. The integral is over \(d^3x'\), and the argument of the delta functions is \(x' - r_0(t')\) with \(t' = t - \frac{(x-x')}{c}\), so it depends on \(x'\) in two ways.

We can get around this difficulty by defining the new variable

\[ \Xi = x' - r_0(t') \quad (x14.5) \]

(the argument of the delta functions), and integrating over \(d^3\Xi\) instead of \(d^3x'\). The volume elements are related by \(d^3\Xi = J d^3x'\) where

\[ J = \begin{vmatrix} \frac{\partial \Xi}{\partial x'} & \frac{\partial \Xi}{\partial y'} & \frac{\partial \Xi}{\partial z'} \\ \frac{\partial \Xi}{\partial x'} & \frac{\partial \Xi}{\partial y'} & \frac{\partial \Xi}{\partial z'} \\ \frac{\partial \Xi}{\partial x'} & \frac{\partial \Xi}{\partial y'} & \frac{\partial \Xi}{\partial z'} \end{vmatrix} \quad (x14.6a) \]

\[ \Xi = (\Xi, \eta, \zeta) \]

So we "only" have to evaluate 9 derivatives. (If we're good at pattern recognition, we can just evaluate 2 of them and fill in the rest by analogy.)
Define $\hat{n} = \frac{R}{R}$. Since $t' = t - \frac{R}{c}$, we have

$$\frac{dt'}{dx'} = -\frac{1}{c} \frac{dR}{dx'} = -\frac{1}{c} \frac{d}{dx'} |x-x'|$$

$$= -\frac{1}{c} \frac{d}{dx'} \left[(x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{\frac{1}{2}}$$

$$= -\frac{1}{c} \left( \frac{1}{2} \right) \left( \frac{-2(x-x')}{|x-x'|} \right) = \frac{n_x}{c} \quad \text{etc.}$$

so

$$\frac{d\mathbf{x}}{dx'} = \frac{d}{dx'} \left[x' - x_0(t')\right] = 1 - \frac{dx_0}{dt'} \frac{dt'}{dx'}$$

$$= 1 - \frac{x_0}{c} n_x$$

$$\frac{d\mathbf{y}}{dx'} = \frac{d}{dx'} \left[y' - y_0(t')\right] = -\frac{dy_0}{dt'} \frac{dt'}{dx'} = -\frac{y_0}{c} n_x$$

etc. Filling in the rest by analogy, we have

$$J = \begin{vmatrix}
1 & \frac{x_0}{c} n_x & \frac{x_0}{c} n_y & \frac{x_0}{c} n_z \\
-\frac{y_0}{c} n_x & 1 & \frac{y_0}{c} n_y & \frac{y_0}{c} n_z \\
-\frac{z_0}{c} n_x & -\frac{z_0}{c} n_y & 1 & \frac{z_0}{c} n_z \\
\end{vmatrix}$$

(same terms cancel.) Now, defining $\beta = \frac{x_0, y_0, z_0}{c}$,

$$J = 1 - \beta \cdot \hat{n} = 1$$

(x14.7)

Substituting (x14.3, 4, 5, 6, and 7) in (x14.2) gives
\[ E(x, t) = \frac{1}{\sqrt{cR}} \int d^3X \left[ \frac{q\hat{n}}{cRK} \delta^{(3)}(X) - \frac{q\beta(t)}{cRK} \delta^{(3)}(X) \right] \]

\[ = \frac{q\beta(t)}{c} \frac{1}{\sqrt{cR}} \left( \hat{n} - \frac{\beta(t)}{\sqrt{cR}} \right) \]  

\[ \text{(x14.8)} \]

Now all that's left is a few time derivatives.
But again we must be careful.

First, \( \left( \frac{\delta t^1}{\delta t} \right)_x = \frac{\delta}{\delta t} \left( t - \frac{R}{c} \right) = 1 - \frac{1}{c} \frac{\delta}{\delta t} \left| x - x' \right| = 1 + \hat{n} \cdot \beta \frac{\delta t^1}{\delta t} \)

\[ \Rightarrow \frac{\delta t^1}{\delta t} = \frac{1 - \hat{n} \cdot \beta}{c} = \frac{1}{k} \]  

\[ \text{(x14.9)} \]

Now, \( \frac{\delta \hat{n}}{\delta t} = \frac{\delta}{\delta t} \left( \frac{R}{c} \hat{n} \right) = \frac{1}{c} \frac{\delta}{\delta t} \left( \frac{\hat{x} - \hat{x}'}{\left| \hat{x} - \hat{x}' \right|} \right) \)

\[ = \frac{1}{R^2} \left[ - (\hat{x} - \hat{x}') \frac{\delta \hat{x}'}{\delta t} - (\hat{x} - \hat{x}') \frac{\delta}{\delta t} \left| \hat{x} - \hat{x}' \right| \right] \]

\[ = \frac{1}{R} \left[ \frac{\delta \hat{x}'}{\delta t} \frac{\delta t^1}{\delta t} + \hat{n} \cdot \hat{n} \cdot \frac{\delta \hat{x}'}{\delta t} \frac{\delta t^1}{\delta t} \right] \]

\[ = - \frac{c}{Kr} \left[ \beta - \hat{n} \cdot \hat{n} \cdot \beta \right] = - \frac{c}{Kr} \beta L \]

and \( \frac{\delta \beta}{\delta t} = \frac{\delta \beta}{\delta t^1} \frac{\delta t^1}{\delta t} = \frac{1}{k} \beta \)

\[ \Rightarrow \frac{\delta}{\delta t} (\hat{n} - \beta) = - \frac{1}{k} \left[ \frac{c}{R} \beta_L + \beta \right] \approx - \frac{\beta}{k} \]

Assuming \( \frac{|\beta|}{|\beta_L|} \gg \frac{c}{R} = \frac{1}{\text{light travel time}} \)  

(x14.10)

The other time derivative we need is
\[
\frac{\partial}{\partial t}(KR) = R \frac{\partial K}{\partial t} + K \frac{\partial R}{\partial t} \\
= R \frac{\partial t}{\partial t} \frac{\partial K}{\partial t} + K \frac{\partial t}{\partial t} \frac{\partial R}{\partial t} \\
= \frac{R}{K} \frac{1}{\partial t} (\dot{\hat{n}} \cdot \dot{\beta}) - \frac{\partial}{\partial t} |\hat{x} - \hat{x}'| \\
= -\frac{R}{K} \hat{n} \cdot \dot{\beta} - \hat{n} \cdot \ddot{\beta} C \\
= -\frac{R}{K} \hat{n} \cdot \dot{\beta} \quad [\text{with same assumption (x14.10)}] \\
\Rightarrow \frac{\partial}{\partial t} \left( \frac{\hat{n} \cdot \beta}{KR} \right) \approx \frac{-KR \frac{\partial \beta}{\partial t} - (\hat{n} \cdot \beta) \frac{\partial}{\partial t}(KR)}{K^2 R^2} \\
= \frac{-\dot{\beta} R + (\hat{n} \cdot \beta) R \hat{n} \cdot \dot{\beta}}{K^2 R^2} \\
= \frac{(\hat{n} \cdot \beta) \hat{n} \cdot \dot{\beta} - K \beta}{K^3 R}
\]

and \((x14.8)\) becomes

\[
E(x,t) = \frac{q}{CK^2 R} \left[ (\hat{n} \cdot \beta) \hat{n} \cdot \dot{\beta} - K \beta \right] \\
= \frac{q}{CK^3 R} \hat{n} \times [(\hat{n} \cdot \beta) \times \dot{\beta}]
\]

Finally, putting back the subscript "vet" and the definition of \(K\), we have the textbook result

\[
E(x,t) = \frac{q}{CR} \left[ \frac{\hat{n} \times (\hat{n} \cdot \beta) \times \dot{\beta}}{(1 - \hat{n} \cdot \beta)^3} \right] \quad (\approx 14.14)
\]
We want \( B = E \) and \( E \times B \parallel \hat{n} \) so we take

\[
\hat{B} = [\hat{n} \times E] \times \hat{r},
\]

so that \( \hat{B} + \hat{r} \parallel \hat{n} \) and \( E \times \hat{B} \times \hat{r} \parallel \hat{n} \).

The angular distribution of the energy flux is

\[
\frac{d\Phi}{d\Omega} = \frac{cR^2}{4\pi} \left| E \times \hat{B} \times \hat{r} \right| \left( \frac{\partial E}{\partial t} \right)_x
\]

where the factor \( \left( \frac{\partial E}{\partial t} \right)_x \) is included to give the energy emitted per unit emission time \( t' \).

Plugging \((14.9), (14.13), \) and \((14.14)\) into \((14.11)\) gives

\[
\frac{d\Phi}{d\Omega} = \frac{q^2}{4\pi c} \left[ \hat{n} \times [(\hat{n} \times \hat{r}) \times \hat{E}] \right]^2 \left( \frac{\hat{n} \cdot \hat{E}}{\hat{n} \cdot \hat{E}} \right)^2
\]

\[(14.38)\]

In the non-relativistic limit \((\beta \to 0)\) this gives

\[
\left. \frac{d\Phi}{d\Omega} \right|_{NR} = \frac{q^2}{4\pi c} \left[ \hat{n} \times (\hat{n} \times \hat{E}) \right]^2
\]

\[= \frac{q^2 |\vec{v}|^2}{4\pi c^3} \sin^2 \Theta
\]

\[= \frac{q^2 |\vec{v}|^2}{4\pi c^3} \sin^2 \Theta
\]

\[(14.39)\]

\((\Theta = \text{angle between } \hat{n} \text{ and } \hat{E})\).

and integrating over solid angle \( d\Omega = 2\pi \sin \Theta d\Theta \) gives

\[
P = \frac{q^2 |\vec{v}|^2}{2c^3} \int_0^{\pi} \sin^2 \Theta d\Theta = \frac{2}{3} \frac{q^2 |\vec{v}|^2}{c^3}
\]

\[(14.22)\]

the Larmor formula.
Jackson (p. 666) derives the relativistic generalization of the Larmor formula:

\[ P = \frac{2q^2}{3c} \gamma^6 \left[ (\hat{\beta})^2 - (\hat{\beta} \times \hat{\beta})^2 \right] \quad (4.26) \]

which is a Lorentz invariant because it comes from the manifestly covariant form

\[ P = \frac{-2q^2}{3 \mu c^3} \frac{dp_x}{d\tau} \frac{dp_y}{d\tau} \quad (14.24) \]

Note that for non-relativistic motion, the second term in the brackets in (4.26) is negligible compared to the first, and \( \gamma \approx 1 \), so (4.26) reduces to (4.22).